The Portfolio Choice Channel of Wealth Inequality*

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Abstract

This paper studies how differences in portfolio choice across households help explain the highly unequal wealth distribution seen in the data. It has been well documented that participation rates are substantially smaller than the ones predicted in standard models of portfolio choice. Also, both participation rates and risky shares are highly increasing in wealth. However, both features are usually absent in workhorse models of wealth accumulation. We introduce portfolio choice and adjustment frictions into an otherwise standard model of household saving behavior. Calibrating it to U.S. household-level data, we show that the model is able to provide a better fit of the wealth distribution while being consistent with well-known facts of households' portfolio choices. In particular, the model explains roughly half of the gap between top wealth shares predicted by traditional models of wealth accumulation (e.g. Aiyagari, 1994) and the data.

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JEL Classifications: D31 G11, E21 G51,

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1 Introduction

Understanding the right-skewed shape of the wealth distribution has been a long-standing question in economic debate. This question dates back to Pareto (1896) and has gained increasing interest given recent trends in wealth inequality in the U.S. (Piketty and Zucman, 2014; Saez and Zucman, 2016). In fact, top 1% of households in the U.S. own 46% of total financial wealth whilst the bottom 50% owns less than 1%.[1]

Differences in households' wealth accumulation can arise in several ways. For instance, they can be produced by different savings rates or different returns on savings across the wealth distribution. The first channel is commonly asserted in popular debate, however recent evidence from Fagereng, Holm, Moll and Natvik (2019) suggests that net savings are rather flat across the wealth distribution.[2] The second channel claims that richer households earn a higher return out of each dollar saved. The latter channel is supported by recent empirical evidence that documents returns to savings increasing in wealth (Fagereng, Guiso, Malacrino and Pistaferri, 2020; Bach, Calvet and Sodini, 2020). In fact, Fagereng et al. (2020) finds that this result holds even within narrow asset classes. These results suggest that there is a *scale dependent* component of wealth accumulation as a whole, as well as an effect on the return on savings within specific assets (*type dependence*).[3]

This paper intends to shed light on the role of households' portfolio choices in explaining wealth inequality. To do so, we present a model that incorporates well-known facts from households' balance sheets absent in workhorse models of wealth accumulation. In particular, we introduce a quantitative two-asset heterogeneous agent model with portfolio adjustment frictions and idiosyncratic shocks to both income and wealth. The model builds up from the traditional Bewley-Huggett-Aiyagari model but with two assets as in Kaplan, Moll and Violante (2018).[4]

A well-known empirical regularity documented in the literature is that both participation rates and shares (out of total wealth) invested in risky assets are substantially smaller than in the workhorse models of portfolio choice (Merton, 1969; Samuelson, 1969). Furthermore, both statistics are highly increasing in wealth. These facts are in stark contrast with the positive and independent of wealth risky asset share predicted in these models. Moreover, these facts are also absent in most models of wealth accumulation.

[1] Source: 2019 U.S. Survey of Consumer Finances.

[2] They study saving rates using administrative records from Norway. Net savings refer to the change in households' net worth holding asset prices constant. Net saving rate refers to net savings out of total income.

[3] Perhaps the most intuitive way to characterize *scale dependence* may be that transactions in financial markets often require transaction fees, which are decreasing in the scale of the investment. Alternatively, *type dependence* may be related to heterogeneous investment opportunities or abilities to process information.

[4] For the solution method we rely on the upwind finite-difference method proposed by Achdou, Han, Lasry, Lions and Moll (2017) to compute both households' policy functions and the stationary distribution.

Therefore, this paper intents to build a bridge between empirical regularities in household finance and quantitative models of wealth accumulation.

Introducing non-convex adjustment costs allows us to replicate these facts by ruling out households at the bottom of the distribution and thus giving rise to portfolio heterogeneity and with that heterogeneous returns. The intuition of how it affects inequality is somewhat similar to other recent models (e.g. Hubmer et al., 2020) where richer households hold riskier portfolios and accumulate more wealth through excess returns of risky assets. The adjustment cost also generates inertia, as portfolio re-balancing is costly, which affects households more exposed to the risky asset (the one subject to the adjustment cost), creating a "saving by holding" effect as in Fagereng et al. (2019).

We find that the calibrated model is able to narrow the gap in top wealth shares by roughly half compared to the frictionless benchmark. We then perform a counterfactual analysis to show the amplifying effect of the *scale dependence* as it interacts with households' precautionary savings. This analysis also allows us to decompose changes in top wealth shares into *luck* and *scale*, where the latter accounts for roughly 90%.[5]

Adjustment frictions are modeled using non-convex adjustment costs as in several papers of the portfolio choice literature. This assumption relies on empirical evidence from Vissing-Jorgensen (2002) that finds trading frequencies increasing in wealth. Moreover, the resulting inertia created by such friction is consistent with findings in the household-finance literature that document the relatively unresponsive behavior of households out of changes in wealth (Brunnermeier and Nagel, 2008; Calvet, Campbell and Sodini, 2009). In the calibrated model, introducing a relatively small adjustment cost (0.75% percent of adjusters stock) is able to generate a large inaction region in households portfolio decisions. Further, we show that much of the inaction comes from both ends of the state space.

Therefore, the contribution of this paper is twofold. First, the quantitative model is able to reconcile roughly half of the gap in top wealth shares while also taking into account stylized facts from households' balance sheets absent in the previous literature. Second, by performing counterfactual exercises, we shed light on how adjustment frictions are key to understanding top wealth shares. As the adjustment cost interacts with households' precautionary motives, small changes to households' savings affect accumulation and thus participation rates, returns to savings and so on.

Literature review. This paper is situated within three strands of literature. First, it is related to a long-standing literature on wealth inequality.[6] This literature dates back to Pareto (1896) with the

[5]We use the term *luck* to refer to different realizations of idiosyncratic shocks across households.

[6]For a thorough review of the literature, see Benhabib and Bisin (2018) and references therein.

notion that the upper tail of the wealth distribution follows a power law.[7] Similarly, random growth theories of the wealth distribution include Wold and Whittle (1957) and Stiglitz (1969). More recently, Piketty and Zucman (2014) present empirical evidence on the evolution of the wealth-to-income ratio in developed economies, while Saez and Zucman (2016) estimates the wealth distribution by using detailed U.S. households balance sheets and tax data.[8]

In the same line, recent empirical work highlights the importance of capital gains for wealth inequality (Feiveson and Sabelhaus, 2019; Fagereng et al., 2019; Kuhn et al., 2020; Martínez-Toledano, 2020). For instance, Fagereng et al. (2019) finds that saving rates across the wealth distribution crucially depend on whether they include or not capital gains.[9] This is explained because richer households own assets that experience persistent capital gains and, unlike predictions from the permanent income hypothesis, hold onto them (which they term "saving by holding"). Further, a recent literature has documented empirically the importance of asset prices in wealth inequality (Fagereng et al., 2016, 2020; Bach et al., 2020). Motivated by these findings, an emerging theoretical literature has started to emphasize the importance of portfolio choice, asset prices and heterogeneous returns to saving in wealth inequality (Favilukis, 2013; Gomez, 2018; Hubmer et al., 2020; Xavier, 2020).

Second, this paper is related to the plethora of research regarding household portfolio choice. This literature was born with seminal contributions of Merton (1969) and Samuelson (1969). Both authors proposed analytic models with closed-form solutions and two sharp predictions: (i) agents should participate at all ages in risky asset markets. (ii) risky share is independent of both age and wealth. However, both predictions are at odds with micro data that consistently (through time and across countries) has documented a surprisingly low participation rate among poor households and a steeply increasing risky share across the wealth distribution.[10] This discrepancy is most likely explained by unrealistic assumptions in the model as frictionless financial markets and the absence of labor income risk.

Thus, several papers, though mostly in life-cycle settings, have attempted to match micro-level estimates by adding more realistic features to the workhorse model.[11] Among the most common features, many incorporate fixed adjustment costs (Duffie and Sun, 1990; Gomes and Michaelides, 2005; Gabaix and Laibson, 2001) and or idiosyncratic labor income risk (see e.g., Chang et al., 2018). Other papers have

[7]A random variable S follows a power law if there exist $k, \phi > 0$ such that $\Pr(S > x) = kx^{-\phi}$ for all x.

[8]Also, a prominent literature (e.g., Gabaix, Lasry, Lions and Moll, 2016) studies quantitative models that attempt to match micro-level estimates on both the thick tail inequality and its speed of change.

[9]Net savings rate follows from the standard definition of consumption plus savings equal income, whilst gross saving rate includes capital gains.

[10]See Mankiw and Zeldes (1991); Wachter and Yogo (2010).

[11]See e.g., Bodie, Merton and Samuelson (1992); Heaton and Lucas (1997); Viceira (2001); Haliassos and Michaelides (2003); Cocco, Gomes and Maenhout (2005); Gomes and Michaelides (2003, 2005); Fagereng, Gottlieb and Guiso (2017) and Chang et al. (2018).

gone further in the attempt to match life-cycle dynamics of portfolio choice by incorporating per-period participation costs and rare disaster events (Fagereng, Gottlieb and Guiso, 2017).[12] This paper takes the basic features of this literature for the construction of the model.

From an empirical standpoint, evidence from Vissing-Jorgensen (2002) on trading frequencies suggests the existence of fixed adjustment costs. Similarly, she finds that small per period adjustment costs may explain the non-participation of half the nonparticipants.[13] Note that beyond direct monetary fees, participation costs in risky assets may be closely related to the opportunity cost (e.g. in terms of labor or leisure) of monitoring and gathering financial information for investment decisions.[14]

Third, this paper intends to contribute to the ongoing strand of literature that incorporates rich sources of heterogeneity and incomplete markets to macro models. In that sense, this paper is closely related to workhorse heterogeneous agent models (Bewley, 1986; Huggett, 1993; Aiyagari, 1994) but extended to a two-asset environment as in Kaplan et al. (2018). Further, this paper takes advantage of the finite-difference method proposed by Achdou et al. (2017) to solve the model and to compute the stationary distribution.

Road map. The rest of the paper is organized as follows. Section 2 presents the main source of data used in this paper, descriptive statistics and some stylized facts. Section 3 introduces the baseline heterogeneous agent model. Section 4 presents the calibration strategy and the main results of the baseline model. Section 5 discusses potential extensions to the baseline model and its implications. Finally, Section 6 concludes.

2 Data and stylized facts

2.1 Data sources and descriptive statistics

This section summarizes the main empirical regularities documented in the data. We use the Survey of Consumer Finances (SCF) for the period 1998-2019 as our main data source. This survey is conducted on a triennial basis and offers a rich classification of households' assets and liabilities, which allows to thoroughly classify assets according to their risk and liquidity.[15]

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[12] They allow for a probability of a negative tail event when investing in stocks to capture the exiting of stock markets after retirement.

[13] Namely, a 50 dollar (2000 USD) per period fee.

[14] There are several arguments in this line, for example, mental accounting, cognitive abilities or other behavioral assumptions (e.g. Christelis et al., 2010), or even fully rational models where households optimal portfolio choice depends on the cost of acquiring/processing information (e.g. Huang and Liu, 2007).

[15] For details on the definition of households assets see Appendix A

However, despite the advantages of survey data to explore households' balance sheets, given the cross-sectional nature of the SCF, this data is subject to several sources of endogeneity that must be taken into account. First, household portfolio allocation is most likely influenced by the business cycle.[16] For example, households may want to liquidate assets to smooth consumption in busts.[17] Second, when pooling data from different surveys to analyze portfolio allocation, the results are subject to cohort effects, as noted by Fagereng et al. (2017). Third, the data structure misses structural changes in variables relevant to household portfolio allocation by treating households from different surveys equally. For instance, Favilukis (2013) argues that the higher participation rates can be explained by a decreasing trend in the participation costs in equity markets. Therefore, this section does not intend to provide causal evidence but to characterize a set of stylized facts to be covered in the quantitative model later.

Table 1 presents descriptive statistics for the main assets in households' balance sheets.[18] The first panel presents total safe assets and the main categories broken by asset class. Likewise, the second panel reports specific risky assets. Following Chang et al. (2018), we split several assets (e.g. mutual funds) according to the risk of the underlying asset. For example, equity mutual funds go to risky assets, bonds mutual funds to safe assets, and combinations of both are divided equally between asset types. The third panel reports total financial wealth according to our baseline definition (See Appendix A). Finally, the fourth panel summarizes net house wealth in the sample.

2.2 Stylized facts on wealth inequality and portfolio choice

As a first motivating fact, we show that wealth inequality has increased over time. Panel (A) of Figure 1 plots the evolution of two common measures of inequality (the top 1% wealth share and the Gini index) for our baseline definition of financial wealth. The data shows a steeply increasing trend over the sample period, with both measures consistently increasing over time. Albeit not explicitly covered in the model, this figure remarks the importance of understanding the drivers of wealth inequality and its dynamic. Another well-known fact is that the distribution of financial wealth is extremely right-skewed. The blue line in panel (A) supports that fact as the top 1% holds more than 45% of the total financial wealth in 2019. Further, panel (B) plots the distribution of financial wealth in the SCF in 50 equal-sized bins. Note that roughly 35% of the households hold less than $6,000$ dollars, while about 12% have a financial wealth larger than $500,000$ dollars.

[16] Though evidence from Brunnermeier and Nagel (2008) and Calvet et al. (2009) suggest that households are somewhat unresponsive to changes in asset prices

[17] Nevertheless, evidence from Kaplan and Violante (2014) suggests that losses from imperfect consumption smoothing are second-order relative to gains from higher-return assets, especially in the presence of transaction costs. See also Cochrane (1989); Browning and Crossley (2001).

[18] Also, Table B.1 in Appendix B.1 presents descriptive statistics for demographic variables.

Table 1: Summary statistics

	Mean	Sd	P10	Median	P90	Part. Rate
Total Safe Assets (S)	112,502	754,336	94	12,838	217,628	0.93
Checking Account	6,423	58,129	0	1,338	11,200	0.84
Savings Accounts	14,809	164,161	0	94	24,617	0.55
Money Market Accounts	5,569	99,903	0	0	1,154	0.12
Cerificates of Deposits	6,874	73,597	0	0	1,817	0.11
Savings bond (safe)	8,001	272,823	0	0	426	0.14
Mutual Funds (safe)	11,185	285,924	0	0	0	0.05
IRA (safe)	17,161	106,690	0	0	22,713	0.19
Pensions (safe)	16,352	91,101	0	0	30,951	0.27
Total Risky Assets (R)	127,304	1,102,643	0	323	205,075	0.51
Brokerage	2,459	112,166	0	0	0	0.02
Stocks	40,557	767,154	0	0	12,315	0.17
Mutual Funds (risky)	26,906	398,752	0	0	3,941	0.11
Savings bond (risky)	1,773	113,179	0	0	0	0.01
IRA (risky)	24,041	143,754	0	0	37,500	0.22
Pensions (risky)	21,536	120,580	0	0	42,646	0.31
Baseline Financial Wealth (S + R)	239,806	1,620,250	105	21,238	437,007	0.94
Net House Wealth	171,922	766,263	0	45,222	367,000	0.68

Notes: Data from the SCF for the period 1998-2019. All variables expressed in 2010 USD and deflected using CPI data from the World Bank's World Development Indicators (WDI). Baseline definition of financial wealth follows Chang et al. (2018). For the specific definition see Appendix A.

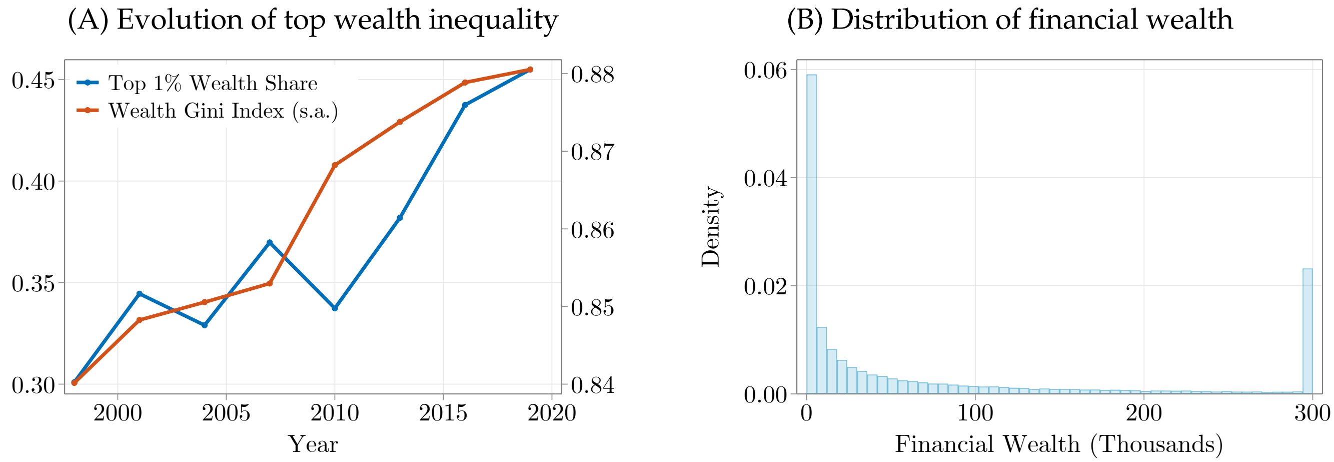
The strikingly uneven distribution of financial wealth may be due to different savings rates or different returns to savings. In the spirit of the growth literature, one might think of both types of accumulation as perspiration and inspiration respectively. Namely, richer households may accumulate more wealth because they save at higher rates (perspiration), or because they allocate resources to higher-return assets (inspiration). Many empirical studies focus on the latter as an important driver for wealth inequality (Piketty and Zucman, 2014; Bach et al., 2020; Fagereng et al., 2020; Kuhn et al., 2020), we take this evidence as an input for our model as there are substantial differences in households balance sheets that have been understudied in the quantitative literature.[19]

To this end, note from Table 1 that participation rates are substantially higher in safe assets and that the median household is absent in most of the asset markets.[20] The lack of participation of median-wealth households suggests the existence of heterogeneity in portfolio choices, sensitive to wealth scale.

[19] Both recent improvements in access to reliable micro data on households' balance sheets and the development of computational methods to solve macroeconomic models with realistic micro foundations are probably at the core of the increasing interest in this channel.

[20] Also note that housing is the main household asset with a mean stock greater than the mean financial wealth.

Figure 1: Financial wealth



Notes: Data from the SCF for the period 1998-2019 in 2010 USD. Both panels use baseline our definition of financial wealth (see Appendix A). Figure B.1 in Appendix B.2 presents the distribution broken by asset class. Panel (A) presents the evolution of the top 1% of wealth shares on the left-hand side axis and the Gini index on the right-hand side axis for each survey. Panel (B) plots the distribution of wealth in our data.

To address this issue, and keeping notation from Table 1, define risky share as

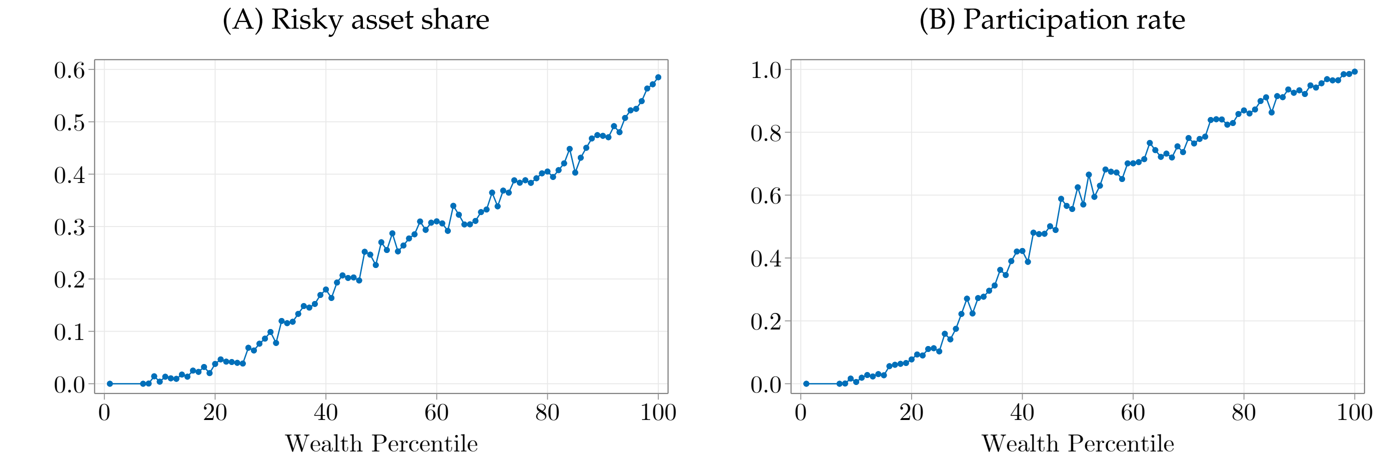
$$\omega = \frac{R}{R+S} \tag{1}$$

Figure 2 plots households' portfolio choices over the financial wealth distribution for our baseline definition of financial wealth. Panel (A) presents the share of financial wealth allocated to risky assets, averaged across percentiles of financial wealth. The figure describes a steeply increasing risky share. In particular, households below the 20th percentile of the distribution hold little to no risky assets on average, while the top decile allocates over half of their savings to risky assets. Panel (B) presents participation rates in risky assets (i.e. share with $R > 0$), which suggests that this heterogeneity is likely to be produced along the extensive margin, as participation is very limited at the bottom of the distribution and almost full at the top. Figure B.2 also shows these are consistent across each survey.

The fact that participation rates are substantially smaller than the ones predicted by standard theoretical models makes room for theories incorporating different frictions on risky asset participation. For instance, households probably face large costs when monitoring risky asset investments and are exposed to big losses due to delayed adjustment (Duffie and Sun, 1990; Lynch, 1996; Gabaix and Laibson, 2001).[21] Similarly, many countries charge operational fees for buying/selling assets, which may be relatively larger for poorer households. This paper builds on this motivation for the quantitative model to be presented in the subsequent section.

[21] For example, spending t hours monitoring investment at an hourly wage w, has an opportunity cost $w \times t$.

Figure 2: Portfolio choices across the wealth distribution



Notes: Risky Share defined in Appendix A. Participation rate in the risky asset is defined as $\mathbf{1}(R > 0)$. Each connected point represents the average risky share/participation rate for households in each percentile. Wealth distribution is computed using our baseline definition of financial wealth.

Both participation rates and the risky share are relevant actors in households' wealth accumulation if average returns differ across asset classes. A simple back-of-the-envelope example allows us to illustrate this point. Consider an individual that allocates x to a "safe" account and x to a "risky" account, with average returns (yearly) of r^b and r^a respectively, such that $r^a > r^b$. The expected difference between holdings in both accounts over t periods is $x[(1+r^a)^t - (1+r^b)^t]$. If we assume, $x = \$10,000$, $t = 10$ years, $r^a = 8\%$ and $r^b = 2\%$, then the expected difference would be $\$10,000 \times [(1+0.08)^{1}0 - (1+0.02)^{1}0] \approx \$9,400$, i.e., 94% of the original investment.

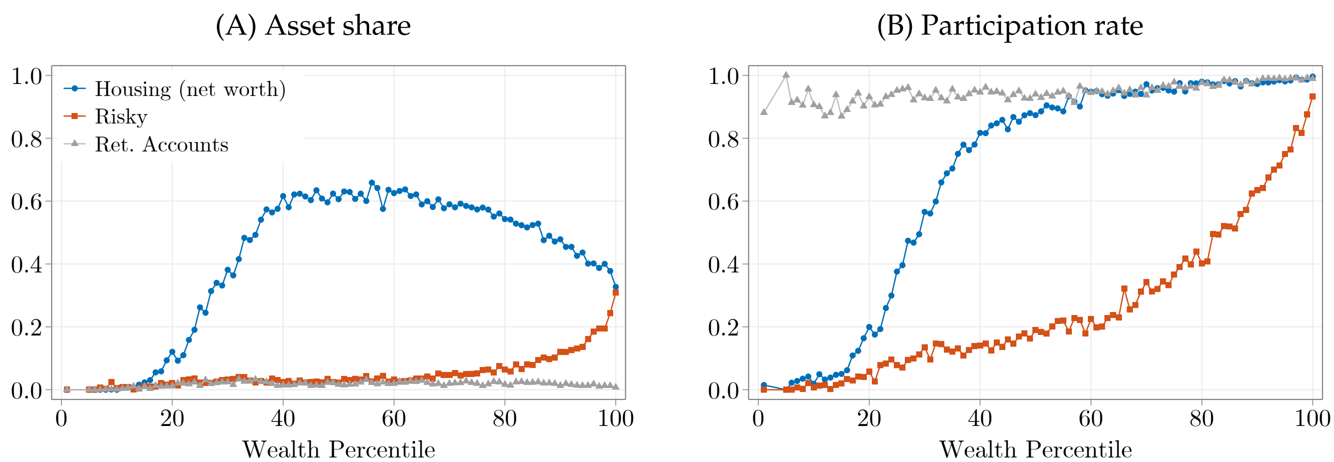
Regarding the definition of financial wealth, this paper excludes from its baseline definition two very important assets in households' balance sheets: private business and housing. On the one hand, we abstract from private business, probably the asset class with the highest returns, as its nature is somewhat different than most investment assets. For instance, households with private businesses may participate in management decisions in the company or have private information related to the performance of the business. This compromises the assumption of exogenous processes for labor income and risky assets in the model below and would require to explicitly incorporate occupational choice, which goes beyond the scope of this paper.[22]

On the other hand, dealing with housing in heterogeneous agent models is not straightforward as it constitutes both a durable consumption good and an investment asset. Laibson, Maxted and Moll (2020), though on a different setting, build a heterogeneous agent model that explicitly incorporates housing in the utility function, playing the double role described earlier. However, even though this feature

[22] However, this feature may be an important driver of wealth inequality at the very top. We consider this a promising avenue for future research. For existing models in this fashion see Quadrini (2000); Cagetti and De Nardi (2006)

would allow us to build a more realistic environment, evidence from Chang et al. (2018) and Fagereng et al. (2019) suggests that neither portfolio choice nor "saving by holding" are exclusively driven by housing. Furthermore, using the baseline definition of financial wealth, and incorporating net house wealth, housing loses its appeal to explain trends in wealth inequality at the upper tail of the distribution. Namely, Figure 3 shows that the housing share peaks somewhere around the median of the wealth distribution and then monotonically decreases. On the contrary, other risky assets combined (stocks, risky bonds, mutual funds, annuities, trusts and brokerage) steeply increase at the top of the distribution.

Figure 3: Specific assets over the wealth distribution



Notes: This figure considers a definition of financial wealth that includes housing for computing shares of each asset, participation rates and the percentiles of the wealth distribution.

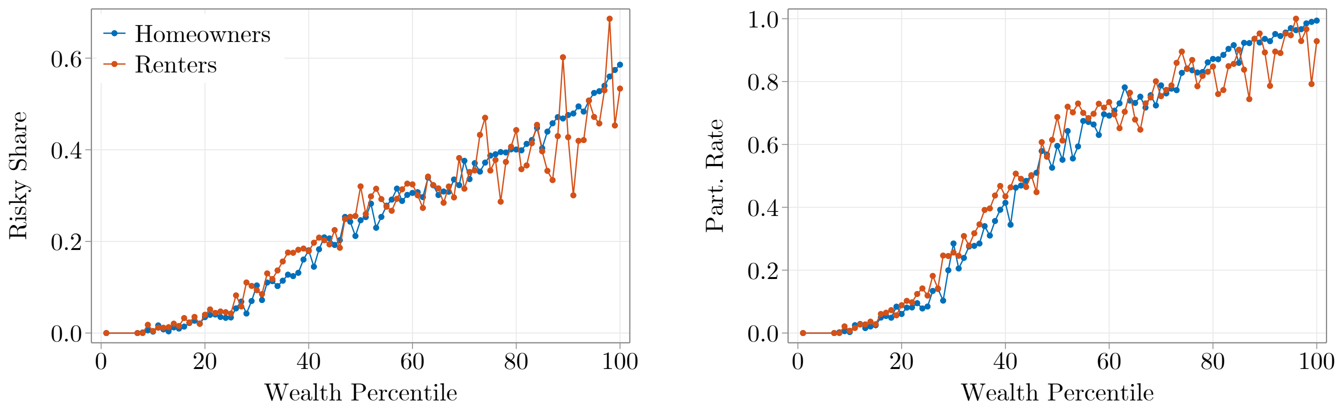
2.3 Sensitivity analysis

In this subsection, we will present risky shares controlling for traditional suspects regarding households' savings decisions. First, we will present descriptive evidence on risky shares for groups commonly related to high/low participation in financial markets. Then we will estimate predicted risky shares by using a simple regression model, controlling for classical determinants of portfolio choice (education, income, age).

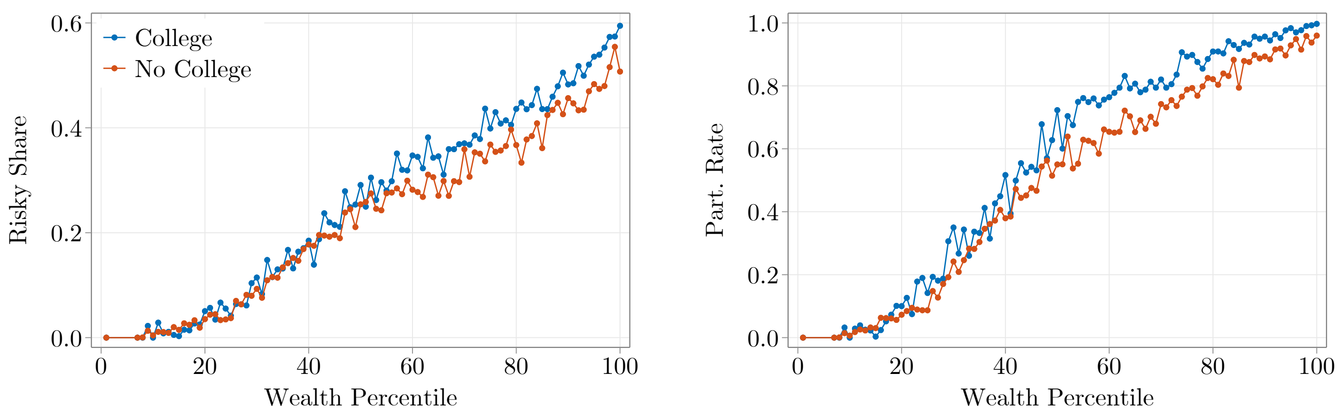
Figure 4 shows that average risky shares and participation rates are somewhat similar across the wealth distribution for both homeowners/renters and college/no college households. Panel (A) shows that despite very few renters at the top of the distribution, their risky share and participation rates follow similar trends than homeowners. Similarly, Panel (B) shows that college graduates hold slightly riskier portfolios and have larger participation rates, though these differences are rather small in magnitude and do not present differential trends over the distribution. These results are consistent with similar exercises conducted by Chang et al. (2018).

Figure 4: Robustness in portfolio decisions

(A) Homeowners versus renters



(A) College versus no college



Notes: Homeowners represent households with housing net worth different than 0. College refers to households with a head with a college degree.

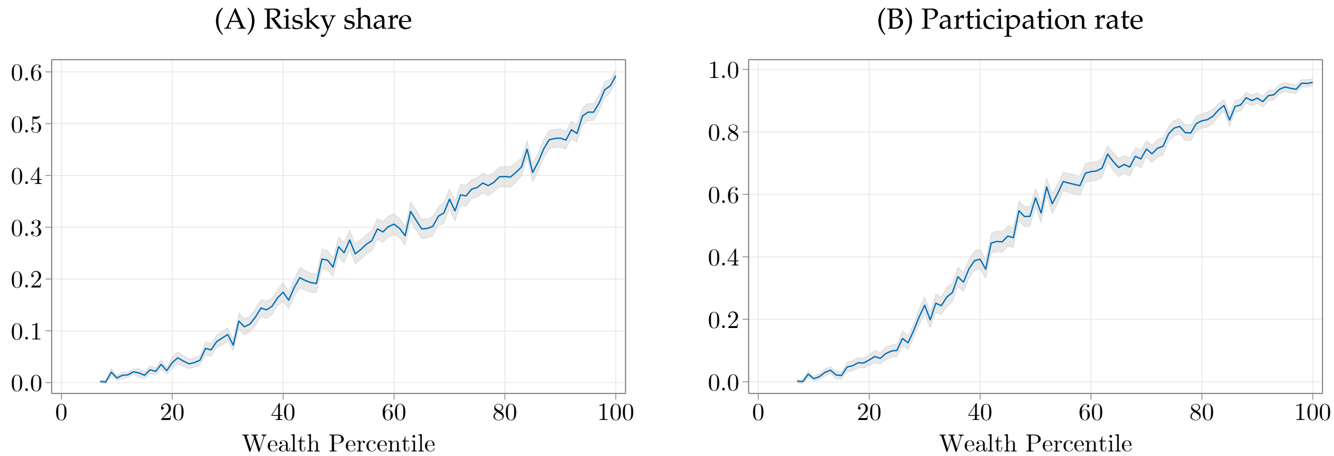
Now for the linear model, we follow Fagereng et al. (2019) and estimate

$$\omega_{it} = \alpha + \sum_{p=2}^{100} \delta_p D_{it,p} + f(\mathbf{x}_{it}) + \mu_t + \varepsilon_{it}, \tag{2}$$

where ω_{it} is the risky share of household i, surveyed on the edition t, $D_{it,p}$ a set of dummy variables for the respective percentiles p, μ_t a year dummy and $f(\mathbf{x}_{it})$ a vector of covariates that include income, homeowner dummy, college dummy, marital status, number of children and a third-order polynomial for age.[23] Figure 5 presents estimates of δ_p for the average risky share and participation rates. Both variables are again increasing in wealth, and with magnitudes similar to the ones shown in Figure 2.

[23]Note that as mentioned, the SCF is not a panel and thus, this is only a cross-sectional regression.

Figure 5: Portfolio choices adjusted by observables



Notes: This figure presents estimates for δ_p in Eq. (2). The blue lines denote point estimates, while the shaded area reports 99% confidence intervals. Panel (A) estimates linear model using risky share as dependent variable. Panel (B) estimates linear probability model using participation in risky assets (i.e., $\mathbf{1}(R > 0)$) as dependent variable.

Hence, there are overall three key stylized facts that will be covered in the next sections. First, the fact that there is a surprisingly small participation rate in risky assets. Second, that both shares and participation rates are steeply increasing across the wealth distribution. Third, the highly unequal distribution of financial wealth. This paper will intend to explain how wealth accumulation is affected by households' portfolio choices. In particular, to what extent does introducing portfolio adjusting frictions that replicate the first two facts contribute to explaining the third.

3 Model

In this section, we describe the quantitative model for the economy. The model describes a household's decision problem with safe and risky assets. Households face both a consumption-saving decision and a portfolio choice, subject to non-convex adjustment costs. The framework is inspired by traditional entry/exit models from firm dynamics literature (Hopenhayn, 1992).[24] It also follows closely the household problem in Kaplan et al. (2018). However, it differs from the latter in the way financial frictions are introduced and in the sense that the two assets are not only different in terms of liquidity but also in the nature of the return process they follow.

Households. Time is continuous and there is no aggregate uncertainty. The economy is populated by a unit mass of infinitely-living households that perceive a utility flow u over consumption $c_t \geq 0$,

[24] Other similar applications of stopping time problems can be found in the international trade literature (Melitz, 2003), investment theory (Dixit, 1989), option pricing (Huang and Pang, 2003), among other.

where u satisfies INADA conditions. Preferences are time-separable and households discount the future at a rate $\rho \geq 0$. Utility is represented by a Constant Relative Risk Aversion (CRRA) function

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma}, & \gamma > 0 \\ \log(c_t), & \gamma = 1. \end{cases} \tag{3}$$

Markets are incomplete and thus households are unable to fully insure against shocks. Households can hold two types of assets. (i) a safe asset b with exogenous return rate $r_t^b(b)$, a borrowing limit $\underline{b} > -\infty$ and a wedge between saving and borrowing $\varpi > 0$.[25] (ii) a risky asset a with stochastic return r_t^a. The risky asset can only be held in non-negative amounts. Deposits or withdrawals in the risky asset are subject to an adjustment cost $\kappa > 0$.[26] This generates an inaction region in households' portfolio decisions. The cost can be interpreted directly as a transactional fee or indirectly as a monitoring cost, mental accounting, among others.[27] Assume for now that the return of the risky asset r_t^a follows a general process of the form

$$\mathrm{d}r_t^a = \mu(r_t^a)\mathrm{dt} + \sigma(r_t^a)dW_t, \tag{4}$$

where W_t is a standard Brownian motion.[28] Labor income follows a two-state Markov process with income streams z_1, z_2, with $z_2 > z_1$. The income process jumps from state z_1 to z_2 with probability λ_1, and from z_2 to z_1 with probability λ_2.[29,30] Households differ on their safe wealth b, risky wealth a and the pair idiosyncratic shocks (z, r^a). At each instant t, the state of the economy is the joint distribution of income z and wealth (a, b).

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[25] That is,

$$r_t^b(b) = \begin{cases} r_t^b, & b > 0 \\ r_t^b + \varpi, & b \in (\underline{b}, 0). \end{cases}$$

Kaplan et al. (2018) refer to this wedge as a "soft borrowing constraint".

[26] Note that this model features non-convex adjustment costs whilst Kaplan et al. (2018) uses a kinked, but strictly convex, adjustment cost function.

[27] See e.g., Gabaix and Laibson (2001); Gomes and Michaelides (2005); Alvarez et al. (2012) among many other. It is also straightforward to allow for a utility cost instead of a monetary cost of adjustment.

[28] A standard Brownian motion is a stochastic process that satisfies $W(t + \Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}$, $\varepsilon_t \sim \mathcal{N}(0, t)$.

[29] With this process we implicitly assume that the income process is independent of the return on the risky asset. Even though in life-cycle settings the covariance term may be important to make households shift towards safer assets as they age, in an infinite horizon setting its importance is of second order. Also, the empirical evidence on the correlation between labor income and risky-asset returns is at least mixed. For instance, Cocco et al. (2005) does not find a significant correlation using the U.S. Panel Study of Income Dynamics (PSID), while Fagereng et al. (2017) finds similar results using administrative data for Norway.

[30] Even though it can be easily extended to a more general process (e.g. diffusion), this setting still has economic interpretation: z_1, z_2 may represent unemployed and employed income streams respectively, with λ_1, λ_2 the job creation and job destruction rates. Moreover, despite being very stylized, this process can reproduce key moments from the income process. In particular, for the benchmark calibration presented below, we discretize an Ornstein-Uhlenbeck (OU) process into a two-state Poisson process that matches the persistence and standard deviation of innovations with traditional estimates in the literature.

Households' asset holdings evolve according to

$$
\begin{aligned}
\mathrm{d}b_t &= (z_t + r_t^b b_t - c_t)\,\mathrm{d}t && (5)\\
\mathrm{d}a_t &= \mathrm{d}r_t^a\, a_t. && (6)
\end{aligned}
$$

Thus, from Eq. (4) and (6), risky wealth evolves according to

$$
\frac{\mathrm{d}a_t}{a_t} = \mu(r_t^a)\mathrm{d}t + \sigma(r_t^a)\mathrm{d}W_t. \tag{7}
$$

Households' value function is given by

$$
v(a,b,z) = \max_{\{c_t\},\tau} \mathbb{E}_0 \int_0^\tau e^{-\rho\, t} u(c_t)\,\mathrm{d}t + e^{-\rho\,\tau}\,\mathbb{E}_0\, v^*(a_\tau + b_\tau, z), \tag{8}
$$

where expectations are taken over realizations of the labor income and the return on wealth, and

$$
v^*(a+b,z) = \max_{a',b'} v(a',b',z) \;\; s.t.\; a' + b' = a + b - \kappa. \tag{9}
$$

From Eq. (8) note that households choose consumption paths, a stopping time τ and a re-balanced portfolio upon adjustment in period $t = \tau$. That is, households choose instantaneous consumption flows but keep their portfolio unchanged until idiosyncratic shocks push them out of their inaction range. When that happens, households pay the adjustment cost κ and re-balance their portfolio by solving Eq. (9).

Altogether, households solve Eq. (8) subject to (5)-(6), the borrowing limit $\underline{b}$, non-negative constraint over consumption and the risky asset and the exogenous processes for z and r^a. This problem can be expressed recursively in terms of the following Hamilton Jacobi Bellman (HJB) equation[31]

$$
\begin{aligned}
\rho\, v(a,b,z) = \max_c \quad & u(c) \\
& + \underbrace{\partial_b v(a,b,z)(z + r^b b - c)}_{\text{Safe Asset}} \\
& + \underbrace{\mu(r^a) a \partial_a v(a,b,z) + \frac{\sigma^2(r^a)a^2}{2}\partial_{aa} v(a,b,z)}_{\text{Risky Asset}} \\
& + \underbrace{\sum_{z' \in Z} \lambda^{z \to z'} \left(v(a,b,z') - v(a,b,z) \right)}_{\text{Labor Income}}.
\end{aligned} \tag{10}
$$

[31] Hereafter we will use short-hand notation $\partial_x v(x) = \frac{\partial v(x)}{\partial x}$ and $\partial_{xx} v(x) = \frac{\partial^2 v(x)}{\partial x^2}$.

with a constraint that

$$v(a,b,z) \geq v^*(a+b,z) \quad \forall\, a,b. \tag{11}$$

The intuition behind Eq. (11) is straightforward. Due to the adjustment cost κ, households do not re-balance their portfolio at every period, that is, $v(a,b,z) > v^*(a+b,z)$. Whenever the value of paying the adjustment cost and re-balancing the portfolio is infinitesimally higher than $v(a,b,z)$, households adjust their portfolio. Hence, $v(a,b,z) = v^*(a+b,z)$ acts as a trigger of the households' portfolio decisions.

Further, Eq. (10) is also subject to a state constraint boundary condition given by

$$\partial_b v(a,\underline{b},z) \geq u'(z + r^b \underline{b}), \tag{12}$$

which is simply a combination of the borrowing constraint in the liquid asset b and the first-order condition in Eq. (10). Suppressing dependence of v in (a,b,z), this problem can be written compactly as[32]

$$\begin{aligned}\min \Big\{ \rho v - \max_c \; u(c) - \mu(r^a) a\, \partial_a v - \frac{\sigma^2(r^a) a^2}{2} \partial_{aa} v - (z + r^b b - c)\, \partial_b v \\ - \sum_{z' \in Z} \lambda^{z \to z'} \left(v(z') - v(z) \right), v - \mathcal{M} v \Big\} = 0,\end{aligned} \tag{13}$$

where $v^* = \mathcal{M}v$ and $\mathcal{M}$ is known as the "intervention operator". This equation is frequently used in mathematics and is called Hamilton Jacobi Bellman Quasi-Variational Inequality (HJBQVI).[33]

In matrix notation, the discretized problem is given by

$$\min \Big\{ \rho \mathbf{v} - u(\mathbf{v}) - \mathbf{A}(\mathbf{v})\, \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \Big\} = 0. \tag{14}$$

Stationary distribution. The evolution of the joint distribution of wealth and income can be described by the Kolmogorov Forward (KF) equation.[34] Define $g(a,b,z,t) \geq 0$ as the density function corresponding to the joint distribution of the states in period t and $s^b(a,b,z)$ as the saving policy function for the safe asset. Then without adjustment, the stationary density satisfies the KF equation[35]

$$\begin{aligned}0 = -\partial_a(\mu(r^a) a g(a,b,z)) + \frac{1}{2}\partial_{aa}(\sigma^2(r^a) a^2 g(a,b,z)) - \partial_b[s^b(a,b,z)\, g(a,b,z)] \\ - \sum_{z' \neq z} \lambda^{z \to z'} g(a,b,z) + \sum_{z' \neq z} \lambda^{z' \to z} g(a,b,z').\end{aligned} \tag{15}$$

32 Appendix D.1 presents in detail the derivation of this equation.

33 See Bensoussan and Lions (1982, 1984); Barles, Daher and Romano (1995); Bardi and Capuzzo-Dolcetta (2008) and Øksendal (2003)

34 Also, often referred to as the Fokker-Planck equation.

35 Achdou et al. (2017) presents an intuitive derivation of the KF equation, though in a simpler setting with one asset and no stochastic returns.

The key advantage of this setting is that the differential operator in the KF equation is the adjoint of the operator in the HJB equation (Achdou et al., 2017).[36] Therefore, the discretized problem is simply

$$0 = \mathbf{A}^T g. \tag{16}$$

However, Eq. (15) is unable to capture adjustment. To incorporate adjustment we need to reallocate mass in the adjustment region to their optimal adjustment targets denoted (a^*, b^*). Luckily, although adding these extra terms mathematically is not straightforward, it can be easily dealt with numerically.[37]

Finally, the stationary distribution $g(a, b, z)$ must satisfy

$$1 = \sum_{z \in Z} \int_{\underline{b}}^{\infty} \int_0^{\infty} g(a, b, z) \, \mathrm{d}a\mathrm{d}b. \tag{17}$$

Hence, the economy can be represented by a system of equations given by (17), the HJB and the KF equations.

4 Baseline quantitative analysis

This section presents the main results from solving the model presented in the preceding section. The model is calibrated at an annual frequency using a combination of externally and internally calibrated parameters, targeting specific moments from the data.

4.1 Calibration

Labor income. There are four parameters to calibrate for the income process: $\{z_1, z_2, \lambda_1, \lambda_2\}$. These parameters are calibrated as follows. We assume discrete time log income follows an AR(1) process of the form

$$\log(z_t) = \varphi_z \log(z_t) + \nu_t. \tag{18}$$

[36] The adjoint operator is a generalization of a matrix transpose for infinite-dimensional operators. Formally, defining the inner product of two functions v and p as

$$\langle v, p \rangle = \int_0^{\infty} v(x) p(x) \mathrm{d}x,$$

the adjoint $\mathcal{A}^*$ of an operator $\mathcal{A}$ is given by

$$\langle \mathcal{A}v, p \rangle = \langle v, \mathcal{A}^* p \rangle.$$

Which is analog to the discrete time case where $\mathbf{v} \cdot \mathbf{p} = \sum_{i=1}^{N} v_i p_i$ and $\mathbf{Av} \cdot \mathbf{p} = \mathbf{v} \cdot \mathbf{A}^T \mathbf{p}$. For more detail see Nuño and Moll (2018).

[37] For a theoretical reference see Bertucci (2020). See Appendix C for the numerical approach to deal with this issue.

Following Guvenen, Kambourov, Kuruscu, Ocampo and Chen (2019) and Aiyagari (1994), we set the autocorrelation $\varphi_z = 0.9$ and the standard deviation of innovations $\sigma_\nu = 0.2$. Then, we recover the drift and the diffusion of the Ornstein-Uhlenbeck process (continuous-time analog of the AR(1) process)

$$\mathrm{d}\log(z_t) = -\theta_z \log(z_{t-1}) + \sigma_z \mathrm{d}W_t, \tag{19}$$

where

$$\varphi_z = e^{-\theta_z}, \tag{20}$$

$$\sigma_z = \frac{\sigma_\nu^2}{2\theta_z}(1 - e^{-2\theta_z}). \tag{21}$$

Average income is normalized to one and income states are $-1, +1$ standard deviations from the mean. That is, $z_1 = 0.79$ and $z_2 = 1.21$. Finally, it can be shown that transition probabilities can be obtained from

$$\lambda^{z \to z'} = \left[\frac{\theta_z}{\pi\sigma_z^2\,(1 - e^{-2\theta_z})}\right]^{\frac{1}{2}} \exp\left(\left[-\frac{\theta_z}{\sigma_z^2}\frac{(\log(z') - \log(z)e^{-\theta_z})^2}{1 - e^{-2\theta_z}}\right]\right), \tag{22}$$

which yields $\lambda_1 = \lambda_2 = 0.25$.

Assets. We use a rather standard calibration for the return on wealth. Namely, the process for the returns of the safe and risky assets are calibrated following Gomes and Michaelides (2005). In particular, the annual return of the safe asset r_b is set to 2 percent whereas the drift and the diffusion of the risky asset are constant and set to $\mu = 0.06$ and $\sigma = 0.18$. That is, the average excess of return is 4 percent annually.

The borrowing constraint $\underline{b}$ is set to 1 times the average annual income and the annual wedge is set to $\varpi = 0.06$ as in Kaplan et al. (2018).

Adjustment cost. The non-convexity of the adjustment cost is arguably the most important driver of portfolio choice and wealth inequality in the model. Nevertheless, given the lack of evidence of transaction costs, there is no direct benchmark from the literature. Hence, we internally calibrate this parameter to match the participation rate in the risky asset from the data.

This approach allows to incorporate the extensive margin of asset accumulation, a feature absent in most of the wealth accumulation literature. As it will be shown below, traditional workhorse models that try to replicate the shape of the wealth distribution often require high participation rates, which is in stark contrast with the well-known lack of participation in risky assets from micro data.

Preferences. Preferences take standard values from the literature. The coefficient of relative risk aversion parameter γ is set to 2. The discount rate is set to $\rho = 0.053$.

Table 2: Baseline calibration

Parameter	Description	Value	Source/Target
Households			
γ	Risk aversion	2	Standard
ρ	Subjective discount rate	0.053	Standard ($\beta = 0.95$)
Assets			
$\underline{b}$	Borrowing limit	-1	1 times avg. income
ϖ	Interest rate wedge	0.06	Kaplan et al. (2018)
r^b	Safe asset return	0.02	Gomes and Michaelides (2005)
μ	Risky asset drift	0.06	Gomes and Michaelides (2005)
σ	Risky asset volatility	0.18	Gomes and Michaelides (2005)
κ	Adjustment cost	0.23	Participation rate
Income Process			
z_1, z_2	Income states	$0.79, 1.21$	$\sigma_z = 0.21$, $\varphi_z = 0.9$, $\mathbb{E}(z) = 1$
λ_1, λ_2	Income jumps	$0.25, 0.25$	Eq. (22)

4.2 The role of the adjustment cost κ

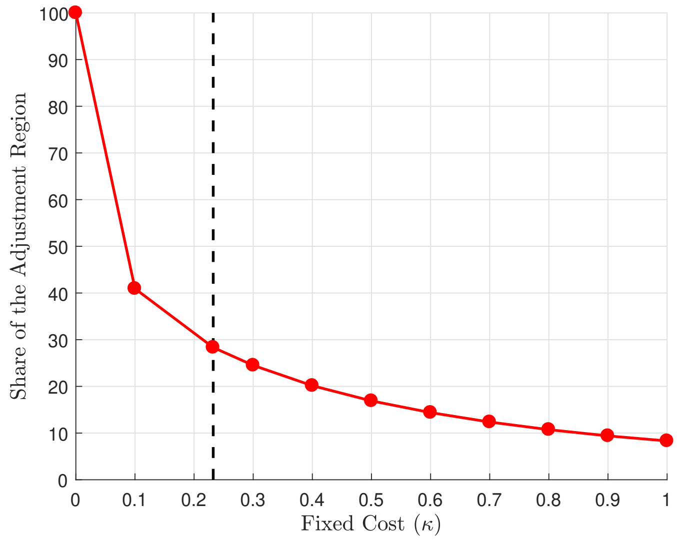
This subsection stresses the importance of adjustment frictions in households' portfolio decisions.[38] The intention of introducing frictions in financial markets is to account for different features that drive households' portfolio decisions, absent in workhorse models in the literature. In that sense, the adjustment cost κ is a reduced form parameter that may represent either direct monetary fees, opportunity costs of planning future investments or more subtle indirect costs like negative utility of acquiring financial skills, information processing costs, mental accounting and so on. Therefore, it is reasonable that the inaction generated in the model is increasing in κ. Given that in order to compute the stationary distribution, mass in the adjustment region must be relocated to the adjustment targets (see Appendix C), we only see this by using households' policy functions. In particular, we compute households stopping time policy function $\tau^*(a, b, z)$ and then measure the size of the adjustment region out of the total state space.

Figure 6 shows the share of the adjustment region for different adjustment costs, including the one used later in the baseline calibration for the model. Clearly, the figure shows that inaction is increasing in the adjustment cost (i.e. adjustment share is decreasing in κ). In fact, the calibrated model only needs a small adjustment cost to match participation rates seen in the data (roughly 51%). Namely, κ represents an average cost of 0.75% of wealth stock for adjusting households, which is in line with previous studies

[38]Unless otherwise noted, we will use the baseline calibration presented in the preceding subsection.

using transaction costs.[39]

Figure 6: Portfolio inertia for different adjustment costs



Notes: This figure reports the share of the adjustment region out of the total state-space L for different adjustment costs. Points in the adjustment region are defined by points where $v(a, b, z) = v^*(a, b, z)$. Vertical line represents baseline calibration for κ.

Furthermore, the model allows us to identify the region of the state space where the adjustment comes from. Namely, Figure 7 presents the adjustment region within the state-space of the economy. The red region denotes adjustment points while the blue region denotes inaction. Note that there is a relatively large inaction range where poorer households (at the bottom left) and richer households (at the top right) do not adjust their portfolio choice. These results are in line with evidence documenting the importance of the extensive margin in portfolio choice (Chang et al., 2018), as well as evidence in saving by holding at the top of the wealth distribution (Fagereng et al., 2019; Kaplan and Violante, 2014).

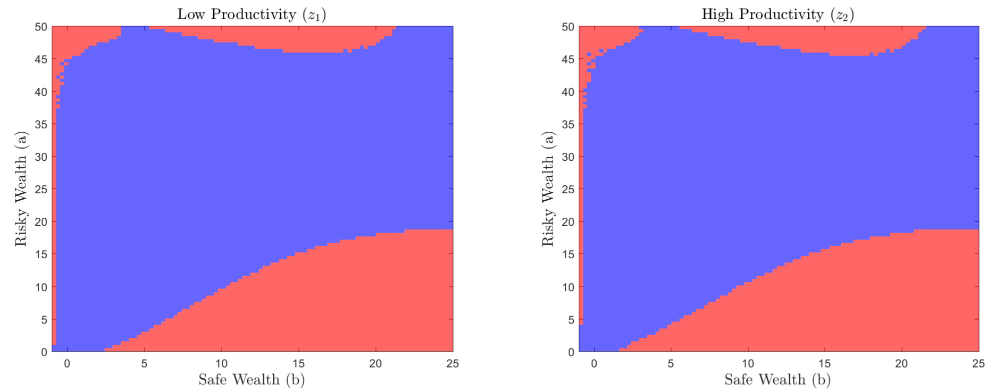
4.3 The Merton model as a useful benchmark

How does the adjustment cost help understand stylized facts in households' portfolio choices? As noted in Section 2, participation rates seen in the data are substantially smaller than the ones predicted in standard portfolio choice models and risky shares are highly increasing in wealth. Given that the adjustment cost κ is calibrated to match portfolio decision along the extensive margin (i.e. participation rate), this subsection intends to see to what extent the model is able to replicate the fact that risky shares are highly increasing in wealth.

To do so, note that the model presented in Section 3 is a generalization of the traditional infinite-

[39] For example, Kaplan and Violante (2014), in a liquid/illiquid two-asset model reports a transaction cost of 0.9% of the stock of illiquid assets for adjusting households. Similarly, Alvarez et al. (2012) reports a 1% adjustment cost on durable goods. Also, Individual Retirement Accounts (IRA) are usually subject to costs and penalties for early withdrawals of 10%.

Figure 7: Decision rules



Notes: The red region denotes adjustment points in the grid while the blue region represents the inaction region. Figure on the left denotes low income (y_1) and the figure in the right the high income (y_2). Figure B.4 in Appendix B.2 reports adjustment targets.

horizon portfolio allocation model introduced by Merton (1969). In fact, the Merton model can be recovered by assuming $\kappa = 0$ and no labor income ($z_1 = z_2 = 0$). In that case, and keeping notation from Section 3, we have that optimal portfolio allocation (hereafter the "Merton rule") yields the following risky share

$$\omega^M = \frac{\mu - r^b}{\sigma_r^2 \gamma}, \tag{23}$$

which is time-invariant and independent of wealth.[40] The intuition is that as there is no other source of uncertainty aside from the return of the risky asset, and no constraint in the risky asset, households optimally allocate a constant share that depends only on the average excess of return, the volatility of the investment and their personal preferences for risk.

Now if households face labor income uncertainty and frictions in saving and re-balancing their portfolio, the risky asset share ω^M is not necessarily optimal. For instance, a poor household will need to pay a large (relative to its wealth) cost to enter the risky asset market, and thus, its optimal share should lay below the Merton rule. Similarly, a rich household, less interested in income fluctuations, may allocate large shares into the risky asset and exploit its expected excess returns. Furthermore, κ prevents those households to adjust towards their optimal portfolio in a frictionless world. More subtly, excess returns are small relative to the adjustment cost for poor households, and labor income risk is more important for their portfolio decision. On the contrary, rich households perceive large flows from saving relative to the adjustment cost and are less concerned about labor income.

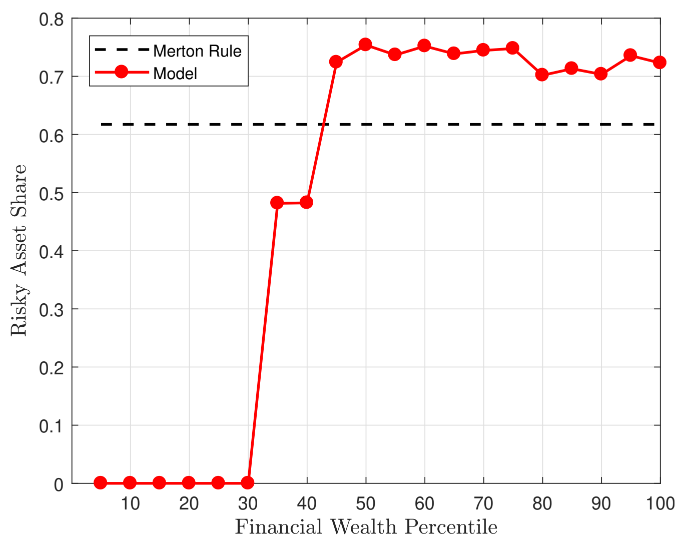
Consistently, Figure 8 presents risky shares across the wealth distribution and compares it to the Merton rule, showing that the risky share predicted from the model is substantially different at both ends

[40] Appendix D.2 solves this model in detail.

of the distribution. Moreover, the model replicates the intuition presented above regarding deviations from the Merton rule. In particular, households below percentile 30 do not hold risky assets and thus, they are either hand-to-mouth or restricted to save only in low-return assets. On the contrary, households at the top of the distribution hold a slightly riskier portfolio than the optimal due to frictions in re-balancing their portfolio. In that sense, since the adjustment cost is fixed, it is relatively smaller for richer households. Therefore, deviations from Merton rule are substantially higher at the lower end of the distribution. Note also that in the model, over some threshold, say the median of the distribution, the risky share is somewhat flat and above the Merton rule.

One important remark of the model is that given discrete adjustments and the fact that all the risky assets are collapsed into one single asset, the replication of risky shares from the data is imperfect. While the data shows an almost linear risky share (though highly increasing), the model predicts non-linear increases. A kinked, though strictly convex, adjustment cost function as in Fagereng et al. (2019) or extending the model to a N asset environment may help to overcome this issue.

Figure 8: Risky share across the wealth distribution



Notes: Each dot in this figure represents the average risky asset share ω for households within groups of 5%. The horizontal line reports the "Merton rule" described in Eq. (23) as a benchmark.

4.4 Stationary distribution of wealth

This subsection presents stationary distribution as the solution of the KF equation introduced in Section 3. Since the model cannot be solved analytically, we employ the already mentioned finite-difference upwind scheme method from Achdou et al. (2017) to solve the model (See Appendix C). One of the biggest advantages of this setting is that allows me to easily compute the stationary distribution of the economy once obtained the transition matrix $\mathbf{A}(\mathbf{v})$. Then with the stationary distribution $g(a, b, z)$, the marginal distribution of both assets in the economy is simply

$$g_a(a) = \sum_{z \in Z} \int_{-\infty}^{\infty} g(a, b, z) \, \mathrm{d}b, \tag{24}$$

$$g_b(b) = \sum_{z \in Z} \int_{-\infty}^{\infty} g(a, b, z) \, \mathrm{d}a. \tag{25}$$

Figure 9: Marginal distribution of assets in steady state

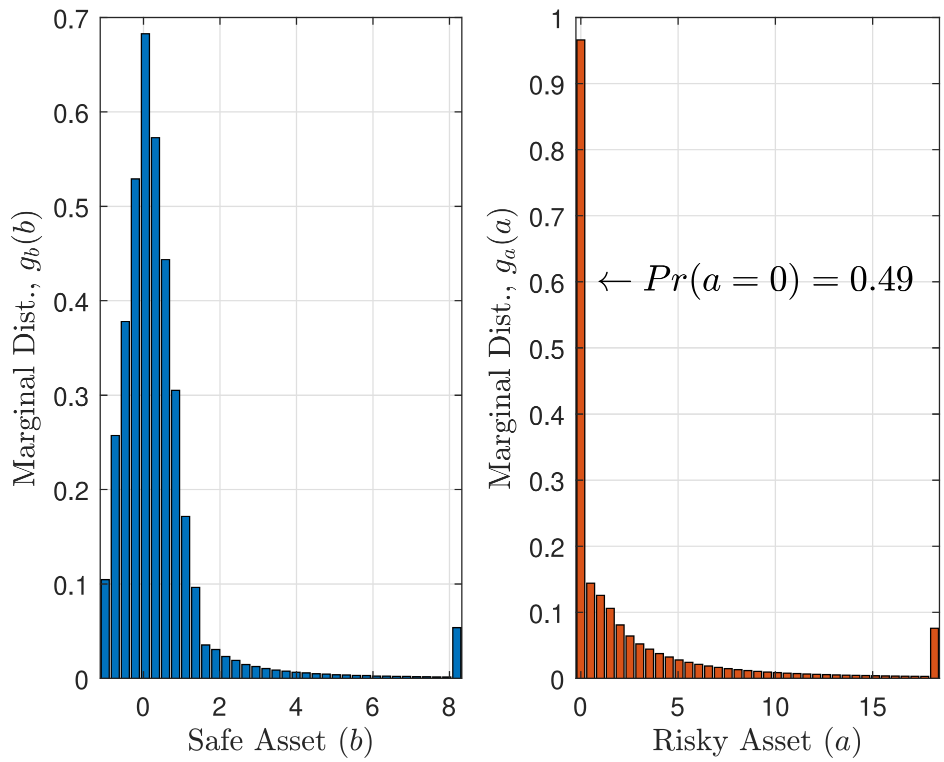


Figure 9 plots the marginal distribution for both assets. This figure shows a highly unequal distribution of wealth, especially regarding risky asset wealth. Note from the panel on the right that the model replicates the participation rate and broadly the shape of the risky asset distribution (see Figure 1). In particular, a great mass of individuals with zero or little risky wealth and a large concentration of wealth from a small group of households.

The main channel through which households diverge on their wealth accumulation is by the combination of *ex-post* realizations of idiosyncratic returns to savings (*luck*) and a *scale dependent* effect generated by the adjustment cost κ. In the spirit of Gabaix et al. (2016), the *scale dependent* effect is produced as portfolio choice (and thus returns) are dependent on wealth. Intuitively, low-wealth households are unable to afford the cost to enter risky asset markets. Further, these households are more sensitive to labor income shock as it represents a higher share of their net worth. Conversely, rich households have incentives to keep their risky assets as selling them for larger consumption streams would require paying the adjustment cost. Also, as they are less sensitive to labor income shocks, they prefer to hold large amounts of the high-return asset, despite fluctuations in their labor income.

4.4.1 Two-asset Aiyagari model as a useful benchmark

To show to what extent the model presented in this paper contributes to the literature we take as baseline the model of Aiyagari (1994) and include a portfolio allocation problem *à la* Merton (1969).[41] That is, heterogeneous households subject to idiosyncratic labor income risk and incomplete markets choose consumption paths and their portfolio *without* adjustment frictions. Assuming the same processes for labor income and stock returns, this model is simply the particular case of $\kappa = 0$ in which all differences in accumulation are generated by different realizations of the idiosyncratic shocks.

Despite not targeting explicitly any moment of the wealth distribution, the model provides a reasonable fit of both the top shares and the bottom end of the distribution. Table 3 reports wealth shares obtained with the baseline model and counterfactual shares obtained by assuming $\kappa = 0$, under the calibration presented in Section 4.1. The results show that the model is capable of providing a better fit throughout the entire wealth distribution. On the bottom end, the share of the bottom 50% is slightly smaller than in the data, though this can be improved by calibrating the borrowing limit to match this moment.[42] On the other end of the distribution, the top shares predicted in the model (top 10%. 5% and 1%) are smaller, though substantially closer than the two-asset Aiyagari benchmark under the same calibration. In particular, the model is able to narrow the gap between the benchmark model and the data in 54% for the top 10% share, 49% for the top 5% share and 41% for the top 1% share.

Another advantage of the model presented in this paper is that it takes into account a well-known fact in the household finance literature absent in most models of wealth accumulation: the surprisingly small participation in risky assets. For instance, the Aiyagari model with portfolio choice and no adjustment frictions is able to match top shares of the wealth distribution under specific calibration, however, it assumes a participation rate substantially greater than the one seen in the data.[43] On the contrary, our model explicitly targets participation share when calibrating the adjustment cost.

[41] Achdou et al. (2017) term this model as "Aiyagari Model with Fat-tailed Wealth Distribution". We will use this name to refer to this benchmark hereafter. For more detail see their online appendix in `https://benjaminmoll.com/papers/`.

[42] Table B.2 jointly calibrates the adjustment cost and the borrowing limit to match the participation rate and the bottom 50%. The results are overall very similar, though with a bottom 50% closer to the data estimates.

[43] Achdou et al. (2017) show that under reasonable assumptions on the risky asset process, the stationary wealth distribution follows an asymptotic power law, where $1 - G(a) \sim ma^{-\zeta}$ with tail exponent

$$\zeta = \gamma \left(\frac{2\sigma^2(\rho - r^b)}{(\mu - r^b)^2} - 1 \right),$$

Table 3: Wealth inequality

Measure	Data	Baseline Model	Fat-tail Aiyagari (1994)
Top 1%	37.5	22.2	11.5
Top 5%	64.6	49.6	35.2
Top 10%	77.8	66.1	52.6
Middle 40%	19.5	33.8	38.3
Bottom 50%	1.0	0.1	9.2

Notes: Estimates of Wealth inequality consider baseline Financial Wealth definition and calibration from Section 4.1.

4.4.2 The amplifying effect of adjustment costs in wealth inequality

Given the stopping time element generated by the adjustment cost, small structural changes in the parameters may have substantial effects on the long-run wealth distribution. For example, small changes in the income process may increase (reduce) precautionary savings and the adjustment cost may amplify (dampen) the effect of wealth inequality. To illustrate this intuition, consider a permanent increase in labor income risk. Due to precautionary motives, higher wage risk will enhance savings, especially from the poor and less-insured households, increasing wealth accumulation at the bottom and thus participation rates, returns to savings and so on. Hence, higher wage risk should reduce top shares, and this effect is likely to be amplified by κ.[44] It is easy to see that the same intuition also holds in the other direction. A decrease in labor income risk will induce higher consumption paths, however rich households, more exposed to the adjustment cost and less sensitive to labor income, will be more reluctant to increase consumption, given that it requires them to pay κ for the withdrawal. This will generate inertia at the top of the distribution.

To test this hypothesis, we take the baseline model, assume a permanent reduction in the standard deviation of log-income innovations of 10% ($\sigma_\nu = 0.18$) and discretize again the income process into a two-state Poisson process as in Subsection 4.1. Then we do the same for the counterfactual case in which there is no adjustment cost (i.e. $\kappa = 0$) and solve both models computing top wealth shares from the resulting stationary distributions. Table 4 reports the results of this exercise. Comparing percentage changes in top shares on both models, it is clear that the amplifying effect of κ is large, with an overall effect on top shares roughly 9-10 times greater than the counterfactual model. Moreover, the effect is quantitatively large by itself. Small changes to the underlying parameters can generate a highly unequal

which calibrated to U.S. data ($\zeta \approx 1.5$) provides a good fit for the wealth distribution, though predicts almost a complete participation in risky assets.

[44] Hubmer et al. (2020) provides a similar intuition to argue the decreasing effect of wage risk in wealth inequality.

wealth distribution through this channel. In particular, this arguably small change to the income process allowed us to get substantially closer to the top shares seen in the data (see Table 3).

Table 4: The amplifying effect of adjustment costs top shares

	Baseline			Fat-tail Aiyagari (1994)		
	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change
Top 1%	22.2	33.9	52.70	11.5	12.2	6.09
Top 5%	49.6	64.1	29.23	35.2	36.6	3.98
Top 10%	66.1	80.1	21.18	52.6	53.6	1.90

Notes: Estimates of Wealth inequality consider baseline Financial Wealth definition. The panel on the left denotes the baseline model presented in Section 3. The panel on the right reports results using a counterfactual model with $\kappa = 0$. Calibration remains unchanged from Subsection 4.1 except for the decrease in wage risk in columns 2 and 4 ($\sigma_\nu = 0.18$).

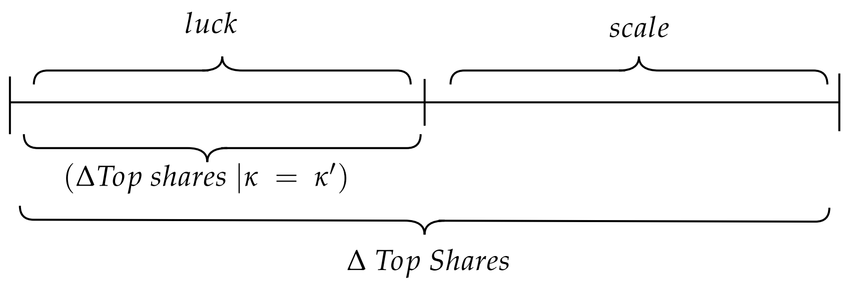
4.4.3 Decomposing top shares

Intuitively, in the benchmark with no adjustment costs (i.e. $\kappa = 0$), all differences in wealth accumulation are given by *ex-post* realization of the idiosyncratic shocks (*luck*), either in the return to savings or in labor income. On the other hand, our baseline model incorporates a *scale dependent* component that deepens wealth inequality. Namely, richer households hold riskier portfolios and exploit positive excess returns. However, both effects are not additively separable. The *luck* component in the return to savings depends on the participation decision and thus on the *scale* component generated by κ.

One way to overcome this issue is to again use the structural change to the income process introduced in Subsection 4.4.2, but now re-calibrate κ to match the participation rate given the new income process. This allows me to isolate the *luck* component of wealth shares and with that decompose, in the lens of the model, top wealth inequality into *luck* and *scale dependence*. Namely, the difference between top shares with and without adjusting κ is the *scale* component of top shares. This is because when calibrating the adjustment cost to match the participation rate given the new income process, we are matching the scale component of wealth accumulation with the one before the shock. Denoting the new adjustment cost $\kappa = \kappa'$, Figure 10 shows how the change in wealth accumulation at the top can be decomposed into the *scale* and *luck* component.

Table 5 presents the results of this decomposition. The first three columns are identical to the ones in Table 4, while the last two columns report the result from decomposing top shares change. The results show that the *scale* component of wealth accumulation accounts for over 88% of the change in top shares predicted by the model. Because in the baseline model households below a certain wealth threshold do not participate in risky asset markets, returns to savings at both ends of the distribution differ substan-

Figure 10: Top share decomposition



tially. Furthermore, as argued earlier, the effect of structural changes to the parameters is amplified by the adjustment cost which fuels top share inequality and thus explains why the *scale* component is the main driver of wealth inequality in the model.

Table 5: Top wealth decomposition

	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change	% scale	% *luck*
Top 1%	22.2	33.9	52.7	88.0	12.0
Top 5%	49.6	64.1	29.2	88.3	11.7
Top 10%	66.1	80.1	21.2	89.3	10.7

Notes: Estimates of Wealth inequality consider baseline Financial Wealth definition and calibration from Section 4.1. The decomposition of top change is computed by comparing the percentage change in column 3 with the change one re-calibrating the adjustment cost to match participation rate.

5 Discussion and extensions

This section explores relevant extensions to the baseline model that may be useful to reconcile with other facts from micro data, address other common sources of wealth inequality, or even policy implications. For instance, the model can be easily extended to incorporate richer return heterogeneity, decreasing relative risk aversion, tax progressivity or quasi-hyperbolic discounting. We will briefly discuss the first two extensions.

5.1 Richer return heterogeneity

The baseline model assumes that the return on savings differs only on the ex-post realization of shocks and the portfolio composition. However, it assumes an *ex-ante* identical process for r^a across households. Nonetheless, there is recent empirical evidence suggesting sizable differences in returns even within narrow asset classes (Fagereng et al., 2020; Xavier, 2020).[45] Therefore, one natural step would be to incor-

[45] Gabaix et al. (2016) terms this feature as *type dependence*

porate *type dependence* into the model. That is, allow both the excess return and the volatility to depend on observable characteristics of households (e.g. wealth). Xavier (2020) takes this approach, though in a one-asset setting, by matching average returns from survey data across the wealth distribution.

Beyond empirical evidence, it is reasonable to expect at least different volatility of returns from a portfolio diversification standpoint. In fact, since the model collapses all risky assets into one single asset, the baseline process for r^a abstracts from imperfect diversification derived from financial frictions. Consider an economy with multiple risky assets, where each transaction requires paying the adjustment cost κ. Intuitively, the adjustment cost prevents households with little wealth to diversify their portfolio when entering the risky asset markets. Households must pay the adjustment cost for buying a first risky asset, and hence there is no portfolio diversification. When households start accumulating assets, they can hold other risky assets and diversify their investment.

Concretely, the model can be easily extended to incorporate richer return heterogeneity and *type dependence* by assuming a more general process for the return r_t^a given by

$$\mathrm{d}r_t^a = \mu(a)\mathrm{d}t + \sigma(a)\mathrm{d}W_t, \tag{26}$$

and a very similar solution method, with the exception that now there are additional "free" parameters to calibrate under the functional form of $\mu(a)$ and $\sigma(a)$.

As an example of this extension, consider the more intuitive case of imperfect portfolio diversification and constant drift (i.e. $\mu(a) = \mu$). Assume also that the volatility of the risky asset decreases exponentially with risky wealth a at a rate ϑ

$$\sigma(a) = \hat{\sigma} e^{-\vartheta a}. \tag{27}$$

Note that there are now two parameters to calibrate $(\hat{\sigma}, \vartheta)$ besides κ. We calibrate the first in order to match the average standard deviation of the risky asset ($\bar{\sigma}$) with the standard deviation on the baseline model ($\sigma = 0.18$). The second is calibrated to match the top 10% wealth share. To calibrate the three parameters, We choose the set of parameters $\hat{\Theta}$ that minimizes the weighted deviation between resulting moments $m(\Theta)$ from the model (computed using the stationary distribution) and the respective targets. That is

$$Q(\Theta) = (m - \hat{m}(\Theta))' \mathcal{W} (m - \hat{m}(\Theta)) \tag{28}$$

$$\hat{\Theta} = \arg\min_{\Theta} Q(\Theta), \tag{29}$$

where $\mathcal{W}$ is the weight matrix, for which we simply assume the identity matrix.

The result of solving Eq. (29) is presented in Table 6, while Table 7 presents the predicted shares of the wealth distribution. Note that despite doing a better job at matching most of the distribution, this approach has problems matching wealth shares at the very top (top 1%). The intuition of this result is that the model *needs* volatility to get some households to draw apart from the rest.

Table 6: Internally calibrated parameters

Parameter	Value	Target	Model
Fixed adjustment cost (κ)	0.19	51.2[a]	49.7
Exponential decay rate (ϑ)	0.01	77.8[b]	74.7
Scale parameter volatility ($\hat{\sigma}$)	0.22	0.18[c]	0.21[d]

[a] Risky asset participation rate.
[b] Top 10% wealth share.
[c] Gomes and Michaelides (2005).

Table 7: Top wealth shares under imperfect diversification

Measure	Data	Baseline Model	Imperfect Diversification
Top 1%	37.5	22.2	19.2
Top 5%	64.6	49.6	54.2
Top 10%	77.8	66.1	74.7
Middle 40%	19.5	33.8	26.8
Bottom 50%	0.98	0.10	-0.2

Notes: Imperfect diversification refers to the case where the risky asset diffusion follows Eq. (27)

5.2 Decreasing relative risk aversion

A very common argument when trying to explain the increasing risky share across the wealth distribution is differences in risk aversion. In particular, that relative risk aversion (RRA) is a decreasing function of wealth. Or perhaps more intuitively, that richer households can "afford to take risks". In that sense, there are two possible ways to generate RRA decreasing in wealth. First, one can exogenously assume preference heterogeneity, for example by assuming a CRRA utility function where γ changes across individuals. The second option is to assume that preferences are represented by a utility function with decreasing RRA.[46]

Regarding the latter, the Stone-Geary utility function slightly changes the typical CRRA function

[46]For a discussion in this matter see Mas-Colell, Whinston, Green et al. (1995) pages 192-193.

by assuming a subsistence level of consumption $\bar{c} > 0$. That is

$$u(c_t) = \frac{(c_t - \bar{c})^{1-\gamma}}{1-\gamma}, \tag{30}$$

where the RRA parameter $\Gamma(c_t)$ is given by

$$\Gamma(c_t) = -\frac{u''(c_t)c_t}{u'(c_t)} = \frac{\gamma c_t}{c_t - \bar{c}}. \tag{31}$$

It is easy to see that $\Gamma'(c_t) < 0$, i.e. that the relative risk aversion is decreasing in wealth. Note also that when $\bar{c} = 0$ we recover the case with constant relative risk aversion.

Both approaches presented above have caveats that are worth mentioning. On the one hand, assuming preference heterogeneity is difficult since we usually do not observe preferences. On the other hand, Stone-Geary utility functions require a positive subsistence level of consumption which is rather uncommon in the literature and also challenging to measure. Either way, the effect of decreasing relative risk aversion depends basically on two opposing forces. As noted by Gomes and Michaelides (2005), more risk-averse households (in this case households at the bottom of the distribution) will be less willing to invest in risky assets. However, greater risk aversion increases saving and thus accumulation at the bottom. Further, as previously noted, this greater accumulation also increases participation in risky assets, amplifying the effect. The net effect in wealth shares will then depend on which force dominates.

For illustrative purposes, we will solve again the model both using a Stone-Geary utility function and incorporating preference heterogeneity separately. For the former, we will now follow Achury, Hubar and Koulovatianos (2012) and assume a monthly subsistence level of consumption of 240 US dollars and an average household size of 2.5. That gives an annual subsistence level of 7,200 dollars. Rounding up average income to 64,000 dollars (See Table B.1), we set $\bar{c} = 11.25\%$ (7200/64,000) given that average income is normalized to one. For the latter, we assume the simplest case where low-income households (which can be interpreted as unemployed) are more risk-averse than high-income households. For simplicity, we set arbitrarily both values to be $\gamma_1 = 1.5$ and $\gamma_2 = 2.5$.

Results are presented in Table 8 shows that overall the results remain unchanged under both ways of introducing decreasing relative risk aversion. The reason for this result is probably related to a similar magnitude between the two opposing forces described earlier. However, both approaches and their calibration were very stylized and do not intend to provide a final answer on the importance of this channel for wealth inequality.

Table 8: Top wealth shares under decreasing relative risk aversion

Measure	Data	Baseline Model	Pref. Heterogeneity	Stone-Geary
Top 1%	37.5	22.2	23.0	20.3
Top 5%	64.6	49.6	49.4	47.3
Top 10%	77.8	66.1	65.7	63.8
Middle 40%	19.5	33.8	33.9	34.7
Bottom 50%	0.98	0.10	0.5	1.5

Notes: Preference heterogeneity denotes the case where preferences are represented by a CRRA utility, with a parameter γ dependent on the income profile z. Model with Stone-Geary utility function incorporates a subsistence level $\bar{c}$ as in Eq. (30).

6 Concluding emarks

For years it has been an open question what drives wealth inequality and the right-skewed shape of the wealth distribution. However, only recently there have been serious attempts to incorporate richer heterogeneity in households' balance sheets, though usually abstracting from key facts documented in the portfolio choice literature. For instance, several papers have tried to match the top shares of the wealth distribution abstracting from households' portfolio choices or frictions in financial markets. This rules out two key stylized facts in households' balance sheets: (i) the surprisingly small participation rates in risky assets and (ii) the steeply increasing risky asset share.

To this end, this paper intended to provide a connection between portfolio choice and wealth inequality by introducing a second (risky) asset and frictions in its adjustment into an otherwise standard wealth accumulation model (e.g. Aiyagari, 1994). Therefore, we first presented well-known facts in the portfolio choice literature and the wealth distribution from the survey data. With that key moments, we calibrated the model to match micro estimates and compute the stationary distribution.

Once computed the stationary distribution, we compared the wealth shares predicted by the model and the ones predicted in the benchmark case with no adjustment cost with estimates from the data, showing that under the same calibration, the model can explain roughly half of the gap in top wealth shares from the benchmark model, without explicitly targeting any parameter of the distribution. Moreover, we showed that the introduction of the adjustment costs amplifies wealth inequality through a *scale dependent* effect on the return to savings which explains most of the changes to the shares at the top.

Finally, we proposed some interesting extensions to the baseline model that may be able to tackle recent empirical evidence on return heterogeneity within asset classes, incorporate imperfect portfolio diversification, or even explore the role of relative risk aversion across the wealth distribution. However, both the baseline model and the extensions abstracted from general equilibrium implications by assuming exogenous processes for the labor income and the assets returns. Also, there is no attempt to microfound the adjustment cost. We consider these issues fruitful areas for future research.

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The Portfolio Choice Channel of Wealth Inequality

Online Appendix—Not for Publication

Mauricio Calani Lucas Rosso

A Data generation and definitions

Survey of Consumer Finances. Households balance sheet data is collected over 8 editions of the Survey of Consumer Finances (SCF) for the period between 1998-2019 (both included). The SCF is a cross-sectional survey of U.S. households conducted on a triennial basis. The survey provides detailed information on households' assets, income and demographic characteristics. Roughly 6,500 households are randomly selected on each survey, with the intention to represent families from all economic strata. The selection of households is designed by using a multistage area-probability design and tax data. The latter source is designed to disproportionately select households relatively wealthy, and usually under-represented in survey data.

For the construction of the variables, we follow closely the definition of Chang et al. (2018).[47] The variables generally remain across surveys but changes to the variables within assets will be mentioned when relevant.

Checking Accounts, Money Market Accounts: The variables X3506, X3510, X3514, X3518, X3522, X3526 report the amount of money in (up to) six different accounts. The respondent answers in variables X3507, X3511, X3515, X3519, X3523, X3527 whether this account corresponds to a checking account or a money market account.

Savings Account: Savings accounts are defined as the sum of the variables X3804, X3807, X3810, X3813, X3816 for the surveys 1998 and 2001 and X3730, X3736, X3742, X3748, X3754, X3760 from the survey of 2004 onwards.

Certificates of Deposit: The variable X3721 provides the money amount the respondent has in certificates of deposit. Only certificates of deposit that belong to someone related to the household are considered (X7620 < 4).

Saving Bonds: Two categories are defined for bonds. Safe bonds consider the sum of variables X3902 (money in U.S. government saving bonds), X3908 (face value of government bonds) and X3910 (money in state and municipal bonds). Risky bonds consider variables X3906 (face value of mortgage-backed bonds), X7634 (face value of corporate bonds) and X7633 (face value of foreign bonds).

Life Insurance: Variable X4006 gives the monetary value of life insurance policies while variable X4010 is the amount currently borrowed using these policies. We define life insurance as the maximum between X4006 - X4010 and zero.

Miscellaneous Assets: Amount of money owed by friends, relatives or others, money in gold or

[47] Also, the replication materials on this paper made easier the processing of surveys in the period 1998-2007.

jewelry and others. Variables X4018, X4022 and X4030 represent money owed to the respondent and X4032 money owed by the respondent. Miscellaneous assets are defined as the difference between the former three and the last variable X4032.

Brokerage Accounts: X3930 gives the dollar value of all the cash or call money accounts at a stock brokerage and X3932 is the current balance of margin loans at a stock brokerage. Brokerage account is defined as the former minus the latter variable.

Mutual Funds: Variables X3822, X3824, X3826, X3828, and X3830 provide the total market value of mutual funds grouped by stocks, tax-free bonds, government-backed bonds, other bonds, and combination funds respectively. Safe mutual funds are defined as $X3824 + X3826 + X3828 + 0.5 \times X3830$ while risky mutual funds consist of $X3822 + 0.5 \times X3830$

Publicly Traded Stocks: The variable X3915 gives the market value of US stocks owned by the respondent, and variable X7641 the market value of stocks of companies outside the US. Stocks are defined as the sum of both variables.

Annuities: The variable X6820 gives the dollar value of annuities for the 1998 and 2001 surveys and X6577 afterward. Only risky (Stocks, mutual funds or real estate) are defined as risky annuities, only safe (bonds/interest, CDS/money market) are defined as safe annuities, and mixed annuities money is allocated in equal parts on each type (risky/safe).

Trust: Variable X6835 (years 1998-2001) and X6587 (years 2004-2019) provide the money amount of assets in a trust. Allocation between risky and safe trusts follow the same criteria as in annuities.

Individual Retirement Accounts (IRA): For the surveys 1998-2001, variables X3610, X3620 and X3630 report the total money in individual retirement accounts. Variable X3631 reports the detail of the investment. Following Chang et al. (2018), money markets, bonds and universal life policy are considered safe, while stocks, real estate, limited partnership investments and brokerage accounts are considered risky. We allocate 2/3 into the safe account when X3631 = 4 (combination of money market, bonds and stocks) and the rest to the risky asset. Similarly, other combinations are divided equally between safe and risky assets.

For the remaining surveys, X6551, X6552, X6553, X6554, X6559, X6560, X6561, X6562, X6567, X6568, X6569 and X6570 report money in individual retirement accounts. Variables X6555, X6563, and X6571 report asset categories. Split accounts are divided equally, interest-earning assets are considered safe and stocks, real estate and hedge funds are considered risky.

Pensions: Variables X4226, X4326, X4426, X4826, X4926 and X5026 provide the total amount of money in pension accounts. We subtract possible loans against these accounts by using the variables: X4229, X4329, X4429, X4829, X4929 and X5029. Categories Savings, 401k, defined contribution plan, money purchase plan, tax-deferred annuity, other types of annuities and others are divided equally between safe and risky pensions. Other pension benefits reported in variables X5604, X5612, X5620, X5628, X5636 and X5644 are also equally distributed across assets. In the 2001 survey variables X5604, X5612, X5620, X5628, X5636 and X5644 are allocated using detail on variables X6491, X6492, X6493, X6494, X6495 and X6496. Interest-earning assets are considered safe, stocks, real estate and insurance risky, and split or other are equally distributed.

For the surveys 2004-2019, pensions are stored in variables X11232, X11532 (only 2004-2007), X11032, X11132, X11332, X11432. Future pension benefits are reported as in the other surveys (variables X5604, X5612, X5620, X5628, X5636 and X5644), with the exception that 2004-2007 surveys exclude X5636 and X5644. Allocation by asset detail is equal to previous editions of the survey.

Housing: Variables X513 and X526 give the value of the land and buildings the respondent (partially) owns. Variables X604 and X614 are the value of the site and the mobile home the respondent owns respectively. Variable X623 is the total value of the home and site if he/she owns both. Variable X716 is the current value of the home/apartment/property the respondent (partially) owns. Variables X1706, X1806 and X1906 (surveys 1998-2016) give the total value of other properties such as real estate investments or vacation houses. Variable X2002 denotes the value of recreational properties and X2012 the remaining value of other properties.

Variables X1409, X1509 and X1619 (until SCF 2010), X1609 (until SCF 2007), X1310, X1328, and X1339 (2013 onwards) provide the value of land contract lending from the respondent. Similarly, variables X1417, X1517 and X1621 (until SCF 2010), X1617 (until SCF 2007), X1318, X1337 and X1324 (2013 onwards) report the value of land contract lending to the respondent.

Variables X805, X905, and X1005 give the value of mortgage debt from the respondent, X1108, X1119, X1130, and X1136 provide the value of lines of credit and X1044 other loans. Finally, housing net worth is defined as the sum of the value of the properties and land contract lending, minus all mortgage debt, lines of credit, land contract borrowing and other loans held by the respondent.

With the previous definitions, we group assets into the following categories:

$$\begin{aligned}\text{Safe Assets } &= \text{Checking Accounts } + \text{ Money Market Accounts } + \text{ Savings Accounts}\\ &+ \text{Certificates of Deposit } + \text{ Safe Saving Bonds } + \text{ Life Insurance } + \text{ Safe Trusts}\\ &+ \text{Miscellaneous Assets } + \text{ Safe Mutual Funds } + \text{ Safe Annuities } + \text{ Safe IRA}\\ &+ \text{Safe Pensions}\end{aligned}$$

$$\begin{aligned}\text{Risky Assets } &= \text{Risky Saving Bonds } + \text{ Brokerage Accounts } + \text{ Stocks } + \text{ Risky Mutual Funds}\\ &+ \text{Risky Annuities } + \text{ Risky Trusts } + \text{ Risky IRA } + \text{ Risky Pensions}\end{aligned}$$

And the baseline definition

$$\omega = \frac{\text{Risky Assets}}{\text{Risky Assets} + \text{Safe Assets}}$$

The baseline definition of Financial Wealth used throughout the text is simply

$$FW = \text{Risky Assets} + \text{Safe Assets} = R + S \tag{A.1}$$

Finally, alternative definitions may include housing (net of mortgage debt).

B Additional tables and figures

B.1 Additional tables

Table B.1: Demographic descriptive statistics from the SCF

	Mean	Sd	Min	Max
Income	63,975.08	149,140.19	0	99,320,000
Age	50.37	17.32	17	95
Children	1.44	1.78	0	10
$\mathbb{1}\{\text{College Degree}\}$	0.39	0.49	0	1
$\mathbb{1}\{\text{Married}\}$	0.50	0.50	0	1

Notes: Data from the SCF for the period 1998-2019 in 2010 US$. Age and children report the respondent age and number of children. $\mathbf{1}\{CollegeDegree\}$ and $\mathbf{1}\{\text{Married}\}$ report whether the respondent has a college degree and if he/she is currently married.

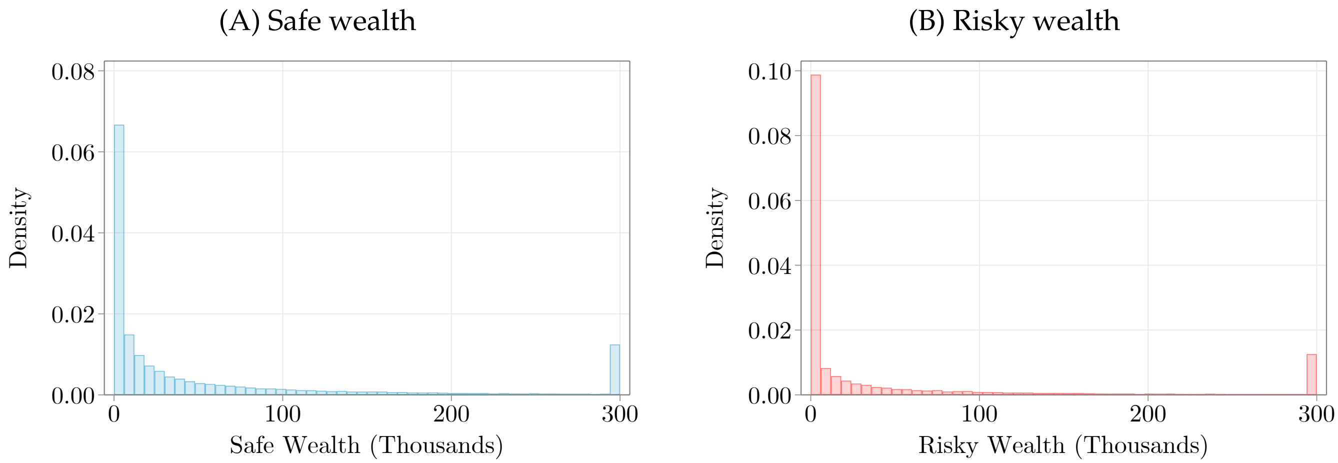
Table B.2: Wealth inequality targeting the bottom of the distribution

Measure	Data	Baseline Model	Targeting bottom 50%
Top 1%	37.5	22.2	22.1
Top 5%	64.6	49.6	49.3
Top 10%	77.8	66.1	65.7
Middle 40%	19.5	33.8	33.6
Bottom 50%	0.98	0.10	0.7

Notes: Estimates of wealth inequality consider baseline definition of financial wealth and calibration from Section 4.1. The first column reports estimates from the data. The second column shows predicted wealth shares from the baseline model. The third column uses the borrowing limit to target the bottom 50% as in Xavier (2020).

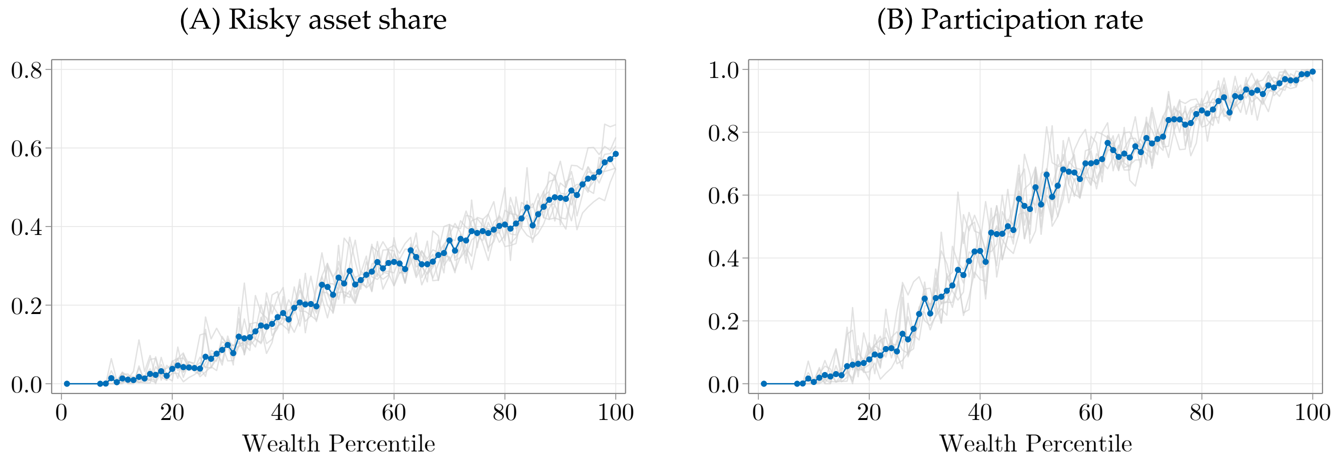
B.2 Additional figures

Figure B.1: Financial wealth distribution by asset class



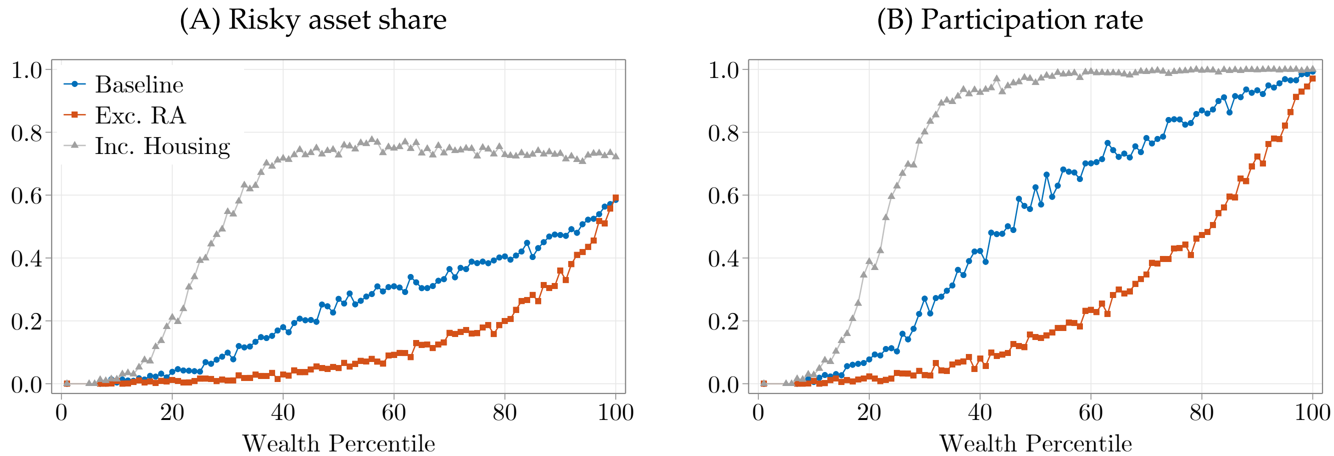
Notes: Data from the SCF for the period 1998-2019 in 2010 thousand USD. Risky/Safe Wealth Definition in Appendix A. Weights in this figure were rounded to the nearest integer.

Figure B.2: Portfolio choices across surveys



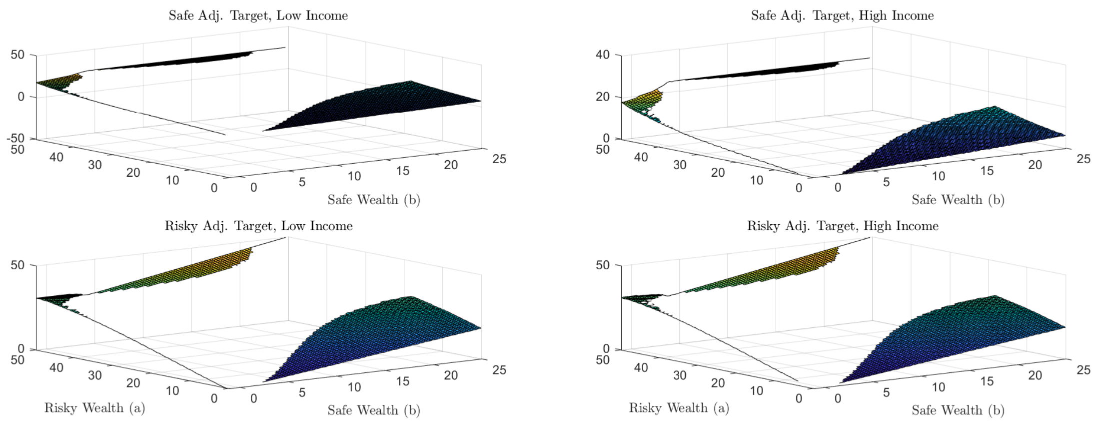
Notes: Risky Share defined in Appendix A. Participation rate in the risky asset is defined as $\mathbf{1}(R > 0)$. Each connected point represents the average risky share/participation rate for households in each percentile. Wealth distribution is computed using our baseline definition of financial wealth. Grey lines denote averages over wealth distribution for each specific survey of the SCF.

Figure B.3: Portfolio choice under different definitions of financial wealth



Notes: Risky Share defined in Appendix A. Participation rate in the risky asset is defined as $\mathbf{1}\{R > 0\}$ Each connected point represents the average risky share/participation rate for households in each percentile. Wealth distribution is computed using the baseline definition of financial wealth (blue), the baseline definition excluding retirement accounts (red) and the baseline definition including housing net worth (green).

Figure B.4: Adjustment targets



Notes: Adjustment targets represent optimal demand for the safe asset b' (first row) and the risky asset a' (second column) for both income profiles, conditional on adjustment.

Figure B.5: Risky asset marginal distribution in steady state

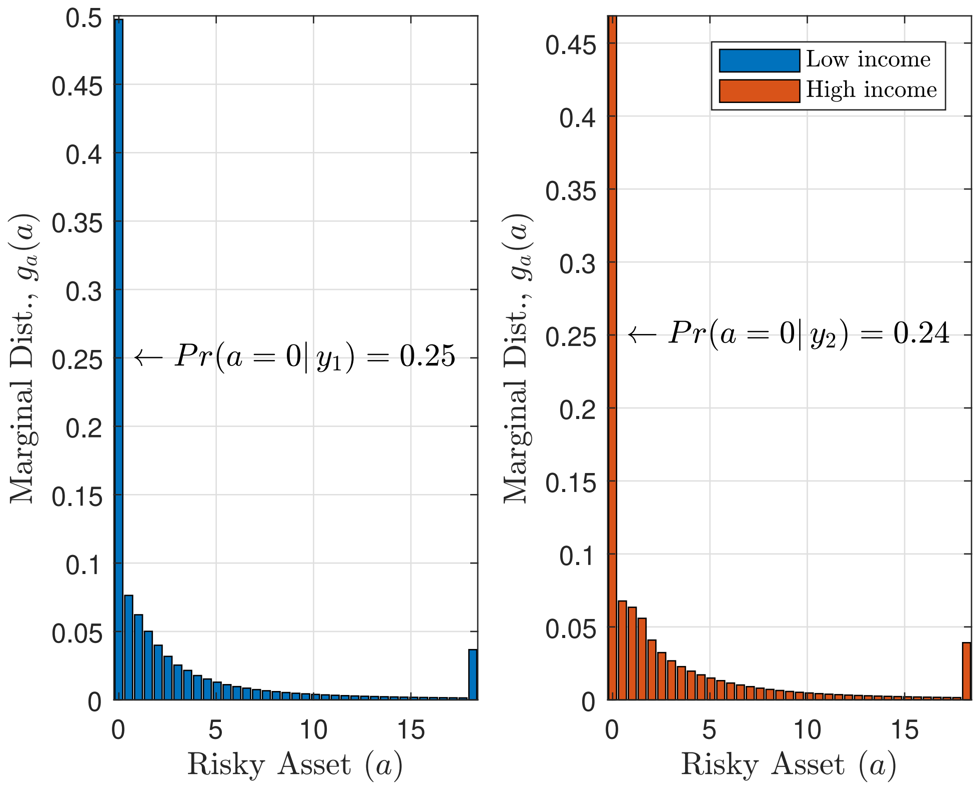
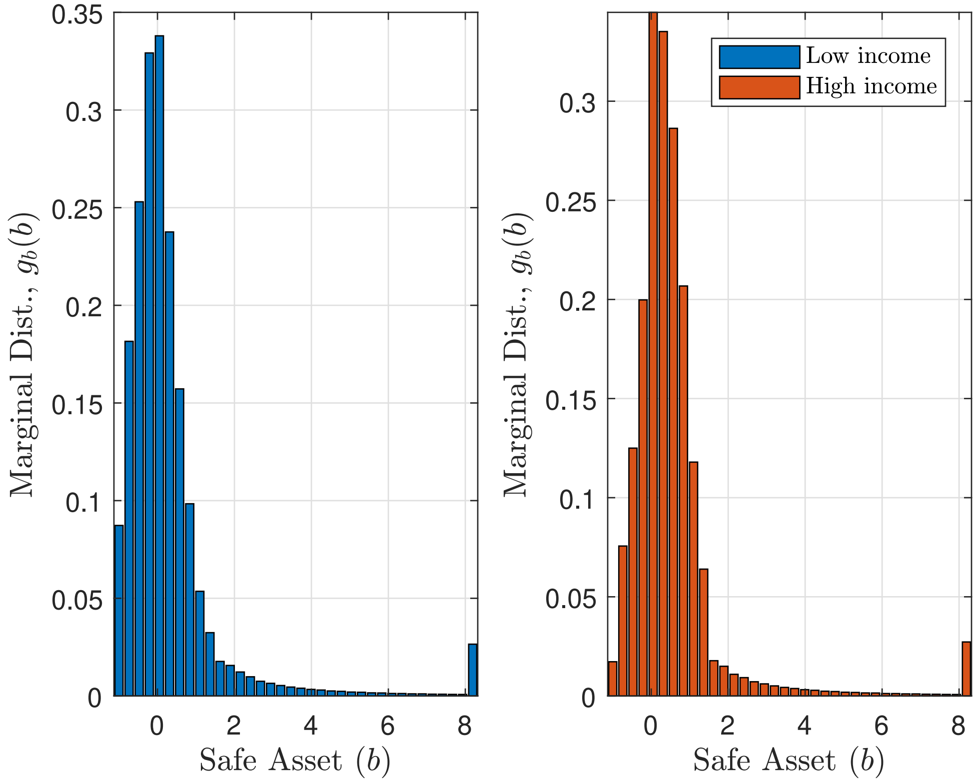


Figure B.6: Safe asset marginal distribution in steady state



C Numerical Strategy

C.1 Solving the HJB Equation

Following closely the finite difference method proposed in Achdou et al. (2017) and building upon Benjamin Moll's public access codes[48], define the equispaced grids $i = 1, \ldots, I$, $j = 1, \ldots, J$, and $z = 1, \ldots, Z$ for the safe asset, risky asset and labor income respectively. Then define $v = v_{i,j,z}$ as the discretized value function. Now define

$$\text{Backward difference: } \partial_{x,B}\, v = \frac{v_k - v_{k-1}}{\Delta x} \tag{C.1}$$

$$\text{Forward difference: } \partial_{x,F}\, v = \frac{v_{k+1} - v_k}{\Delta x} \tag{C.2}$$

$$\text{Central difference: } \partial_{xx}\, v = \frac{v_{k+1} - 2v_k + v_{k-1}}{(\Delta x)^2}, \tag{C.3}$$

as the forward, backward and central derivative approximations for $x = (a, b)$, $k = (i, j)$, and with Δx as the distance between grid points. The idea is then to use backward approximations whenever the drifts of a and b are negative and forward approximations whenever they are positive. The central approximation is used for $\partial_{aa} v$. Now, the discretized version of the household problem can be written as follows

$$\min \left\{ \rho \mathbf{v} - u(\mathbf{v}) - \mathbf{A}(\mathbf{v})\, \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \right\} = 0, \tag{C.4}$$

where $\mathbf{v}$ and $\mathbf{u}(\mathbf{v})$ are the discrete-time value and utility functions over the stacked state space $L \equiv I \times J \times Z$ and $\mathbf{A}(\mathbf{v})$ is a $L \times L$ squared matrix that captures the evolution of the state variables. The algorithm for the solution is the following:[49]

1. As initial guess $\mathbf{v}^0$, use the solution of the no-adjustment case:

$$\rho \mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v})\, \mathbf{v} = 0, \tag{C.5}$$

   i.e. solve the problem when κ is infinite. Note that as the problem is non-linear in $\mathbf{v}$, some iteration will be necessary. The guess for this step will be "staying put"

$$v^0 = \frac{(y + r^b\, b)^{1-\gamma}}{\rho(1-\gamma)} \tag{C.6}$$

[48] https://benjaminmoll.com/codes/.

[49] This algorithm is exactly as in https://benjaminmoll.com/wp-content/uploads/2020/06/liquid_illiquid_numerical.pdf but with the exception that now the process for the return of the risky asset is stochastic, and thus risky wealth evolves slightly differently.

Then, from the FOC w.r.t to c, one can compute

$$c = (\partial_b v_0)^{-\frac{1}{\gamma}} \tag{C.7}$$

for both the backward and forward approximations. Then, it is straightforward to compute backward and forward saving policy functions $s^h = s^h_{i,j,z}$, for $h \in \{B, F\}$. Compute consumption and savings using the upwind scheme. That is, choose backward approximation whenever the drift of b is negative and forward when it is positive.

As risky wealth and labor income evolve exogenously, one can compute directly the evolution of these states using the upwind method. Finally, compute the transition matrix $\mathbf{A}(v)$ and solve (C.5). Iterate until convergence.

2. Given $\mathbf{v}^n$, find $\mathbf{v}^{n+1}$ by solving:

$$\min \left\{ \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} + \rho\, \mathbf{v}^{n+1} - \mathbf{u}(\mathbf{v}^n) - \mathbf{A}(\mathbf{v}^n)\, \mathbf{v}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^*(\mathbf{v}^n) \right\} = 0 \tag{C.8}$$

This step follows Huang and Pang (2003) and its exercise for American options with transaction costs (see section 2.3 on their paper). Eq. (C.4) can be expressed as

$$\Big(\mathbf{v} - \mathbf{v}^*(\mathbf{v})\Big)^T \Big(\rho\mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v})\,\mathbf{v}\Big) = 0 \tag{C.9}$$

$$\mathbf{v} \geq \mathbf{v}^*(\mathbf{v}) \tag{C.10}$$

$$\rho\mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v})\mathbf{v} \geq 0 \tag{C.11}$$

Now denote $\mathbf{z} = \mathbf{v} - \mathbf{v}^*(v)$ and $\mathbf{B} = \rho\mathbf{I} - \mathbf{A}(\mathbf{v})$. Then Eq. (C.10) is simply $\mathbf{z} \geq 0$ and Eq. (C.11) is

$$\mathbf{B}\mathbf{z} + \mathbf{q} \geq 0, \tag{C.12}$$

where $\mathbf{q} = \mathbf{B}\mathbf{v}^*(\mathbf{v}) - \mathbf{u}(\mathbf{v})$. Hence, the problem can now be written in a standard linear complementarity problem (LCP) form

$$\mathbf{z}^T(\mathbf{B}\mathbf{z} + \mathbf{q}) = 0 \tag{C.13}$$

$$\mathbf{z} \geq 0 \tag{C.14}$$

$$\mathbf{B}\mathbf{z} + \mathbf{q} \geq 0, \tag{C.15}$$

which can be easily dealt with LCP solvers.[50]

3. Iterate until convergence.

C.2 Solving the KF Equation

To ease notation, define the discretized variables as $x \equiv x_{i,j,z}$. Then the discretized version of Eq. (15), i.e. the evolution of the joint distribution of states without adjustment, can be written as follows

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}(\mu a g) + \frac{\mathrm{d}^2}{\mathrm{d}a^2}\left(\frac{\sigma^2 a^2}{2} g\right) - \frac{\mathrm{d}}{\mathrm{d}b}(s^b g) - \lambda_k g + \lambda_{-k} g, \tag{C.16}$$

or in matrix notation

$$0 = \mathbf{A}^T \mathbf{g}, \tag{C.17}$$

where $\mathbf{A}^T$ is the transpose of the transition matrix computed in the household's problem (i.e. when solving the HJB equation).

In order to deal with adjustment we need to introduce some additional notation. Define (a_k^*, b_k^*) as the optimal adjustment targets, conditional on adjustment. Then denote $\ell = 1, \ldots, L$ as the stacked state space and the grid point $k^*(\ell)$ reached from point ℓ upon adjustment. Finally, define the inaction region $\mathcal{I}$ as the set of grid points where $v > v^*(v)$.

To introduce adjustment, we need to incorporate a discretization of the "intervention operator" defined in Section 3. To this end, define the binary matrix $\mathbf{M}$ with elements $M_{\ell,k}$, for $\ell = 1, \ldots, L$ and $k = 1, \ldots, L$, given by

$$M_{\ell,k} = \begin{cases} 1, & \text{if } \ell \in \mathcal{I} \text{ and } \ell = k \\ 1, & \text{if } \ell \notin \mathcal{I} \text{ and } k^*(\ell) = k \\ 0, & \text{otherwise} \end{cases} \tag{C.18}$$

This matrix moves mass from the adjustment region to its corresponding adjustment targets. Intuitively it is easy to see that when $\ell \in \mathcal{I}$ and $\ell = k$, $\mathbf{M}$ keeps that element in its same position. However, when $\ell \notin \mathcal{I}$, $\mathbf{M}$ moves that mass to the optimal target $k(\ell)$.

To see how to use $\mathbf{M}$ to solve the KF equation, we follow the "operator splitting method" from Achdou et al. (2017) (see their numerical appendix). Namely, defining $\mathbf{g}^n = \mathbf{g}(t^n), n = 1, \ldots, N$, we use the following algorithm:

[50]For solving the model we use the following Newton-based LCP solver in Matlab: https://www.mathworks.com/matlabcentral/fileexchange/20952-lcp-mcp-solver-newton-based.

1. Given $\mathbf{g}^n$, find $\mathbf{g}^{n+\frac{1}{2}}$ from

$$\mathbf{g}^{n+\frac{1}{2}} = \mathbf{M}^T \mathbf{g}^n \tag{C.19}$$

2. Given $\mathbf{g}^{n+\frac{1}{2}}$, find $\mathbf{g}^{n+1}$ from

$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{AM})^T \mathbf{g}^{n+1} \tag{C.20}$$

3. Iterate until convergence.

This algorithm allows us to tackle two crucial problems when computing the stationary distribution. (i) how to treat density at grid points within the adjustment region. (ii) how to treat grid points in the inaction region in which idiosyncratic shocks from state variables (summarized in the matrix $\mathbf{A}$) move them towards the adjustment region.

Step 1 solves the first problem by moving mass from the adjustment region to their optimal targets in the inaction region. Step 2 solves the second problem by using $\mathbf{AM}$ as the new transition matrix. Recall that $\mathbf{A}$ is a Poisson transition matrix in which rows denote the starting position of a Poisson process and columns the finishing position. Thus, for each row ℓ, $\mathbf{M}$ takes the entries $A_{\ell,k}$ that end up in the adjustment region, and moves them to columns $k^*(\ell)$ corresponding to their adjustment target. Note that $\mathbf{AM}$ is still a valid Poisson transition matrix as M only moves elements from columns and thus, all rows still sum to zero and diagonal elements keep their required properties.[51]

[51] Namely, diagonal elements are still non-positive, while the remaining elements are non-negative.

D Proofs

D.1 Derivation of the HJBQVI

This subsection derives the HJBQVI presented in Eq. (13). To do so, assume a discrete-time environment, with time periods of length Δ and a discount factor given by $\beta(\Delta) = e^{-\rho\Delta}$. As in Section 3, labor income is given by y_1, y_2. Transition probability from income y_j to y_{-j} is given by $1 - p_j(\Delta)$, where $p_j(\Delta) = e^{-\lambda_j \Delta}$ is the probability to keep their current income.[52] Note that the discrete-time analog of the household's problem without adjustment can be summarized by the following Bellman equation

$$v_j(a_t, b_t) = \max_c \; u(c_t)\Delta + \beta(\Delta)\, \mathbb{E}\left[v_j(a_{t+\Delta}, b_{t+\Delta})\right] \tag{D.1}$$

$$s.t. \;\; a_{t+\Delta} = r_t^a a_t \Delta + a_t \tag{D.2}$$

$$b_{t+\Delta} = (y_j + r_t^b b_t - c_t)\Delta + b_t, \tag{D.3}$$

for $j = 1, 2$. Given the probability $p_j(\Delta)$ to keep the current income, we have that Eq. (D.1) can be written as follows

$$v_j(a_t, b_t) = \max_c \; u(c_t)\Delta + \beta(\Delta)\Big\{p_j(\Delta)\mathbb{E}\left[v_j(a_{t+\Delta}, b_{t+\Delta})\right] + \left(1 - p_j(\Delta)\right)\mathbb{E}\left[v_{-j}(a_{t+\Delta}, b_{t+\Delta})\right]\Big\} \tag{D.4}$$

For a small enough Δ we have

$$\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta \tag{D.5}$$

$$\rho_j(\Delta) = e^{-\lambda_j \Delta} \approx 1 - \lambda_j, \Delta \tag{D.6}$$

and thus substituting into (D.4)

$$v_j(a_t, b_t) = \max_c \; u(c_t)\Delta + (1 - \rho\Delta)\Big\{(1 - \lambda_j\Delta)\mathbb{E}\left[v_j(a_{t+\Delta}, b_{t+\Delta})\right] + \lambda_{-j}\Delta\mathbb{E}\left[v_j(a_{t+\Delta}, b_{t+\Delta})\right]\Big\}, \tag{D.7}$$

re-arranging terms

$$v_j(a_t, b_t) = \max_c \; u(c_t)\Delta + (1 - \rho\Delta)\Big\{\mathbb{E}\left[v_j(a_{t+\Delta}, b_{t+\Delta})\right] + \lambda_j\Delta\mathbb{E}\left[v_{-j}(a_{t+\Delta}, b_{t+\Delta}) - v_j(a_{t+\Delta}, b_{t+\Delta})\right]\Big\} \tag{D.8}$$

[52] The derivation presented in this subsection can be easily extensible to a different process for labor income.

Subtracting $(1-\rho\Delta)v_j(a_t, b_t)$, dividing by Δ and taking $\Delta \to 0$ we get

$$\rho v_j(a_t, b_t) = \max_c \; u(c_t) + \frac{\mathbb{E}[\mathrm{d}v(a_t, b_t)]}{\mathrm{d}t} + \lambda_j \left(v_{-j}(a_t, b_t) - v_j(a_t, b_t) \right) \tag{D.9}$$

Finally, we need an expression for the second term on the right-hand side of Eq. (D.9). Given that r^a follows Eq. (4), by Ito's Lemma

$$\begin{aligned}\mathrm{d}v(a_t, b_t) = &\left(\partial_b v(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a \partial_a v(a_t, b_t) + \frac{\sigma^2 a^2}{2} \partial_{aa} v(a_t, b_t) \right) \mathrm{d}t \\ &+ \sigma a \partial_a v(a_t, b_t) \mathrm{d}W_t,\end{aligned} \tag{D.10}$$

taking expectations and noticing that $\mathbb{E}[\mathrm{d}W_t] = 0$

$$\frac{\mathbb{E}[\mathrm{d}v(a_t, b_t)]}{\mathrm{d}t} = \partial_b v(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a \partial_a v(a_t, b_t) + \frac{\sigma^2 a^2}{2} \partial_{aa} v(a_t, b_t) \tag{D.11}$$

Finally, plugging this equation into (D.9) yields Eq. (10). To recover Eq. (13), just note that from the constraint in Eq. (9), $v(a,b) - v^*(a+b) \geq 0$.

D.2 Proof of the Merton Rule

Given no labor income, CRRA utility and no frictions in the financial markets, the household's problem can be written as

$$\begin{aligned} &\max_{\{c_t\}, \omega} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} \mathrm{d}t \\ &s.t. \;\; \dot{a} = a_t \left(\omega (r_t^a - r_t^b) + r_t^b \right) - c_t \\ &\qquad \mathrm{d}r_t^a = \mu \mathrm{d}t + \sigma dW_t, \end{aligned} \tag{D.12}$$

where the HJB equation is given by

$$\rho v(a) = \max_{c,\omega} u(c) + v'(a) \left[a_t \left(\omega(\mu - r^b) + r^b \right) - c \right] + v''(a) \frac{(\sigma a \omega)^2}{2} \tag{D.13}$$

To find the optimal c^*, ω^* we use a guess-and-verify strategy: guess

$$v_0(a) = \frac{B_0^{-\gamma}}{1-\gamma} a^{1-\gamma}, \tag{D.14}$$

with B_0 a constant to be determined. Then $v'(a) = (B_0 a)^{-\gamma}$ and $v''(a) = -\gamma B_0^{-\gamma} a^{-\gamma-1}$. From the FOC in Eq. (D.13)

$$[c]: \; c^{-\gamma} = v'(a) \Longrightarrow c = B_0 a \tag{D.15}$$

$$[\omega]: \; v'(a)(\mu - r^b)a + v''(a)\sigma^2 a^2 \omega = 0 \Longrightarrow \omega = \frac{v'(a)(\mu - r^b)}{-v''(a)\sigma^2 a} \tag{D.16}$$

Using Eq. (D.14) in Eq. (D.16) one can obtain Eq. (23). For completeness, replacing Eqs. (D.15)-(D.16) in Eq. (D.13)

$$\rho \frac{B_0^{-\gamma}}{1-\gamma} a^{1-\gamma} = \frac{(B_a a)^{1-\gamma}}{1-\gamma} + (B_0 a)^{-\gamma} \left[a \left(\frac{(\mu - r^b)^2}{\gamma \sigma^2} + r^b \right) - B_0 a \right] - \gamma B_0^{-\gamma} a^{-\gamma-1} \frac{\sigma^2 a^2 (\mu - r^b)}{2(\gamma\sigma^2)^2} \tag{D.17}$$

Then rearranging terms

$$\frac{\rho}{1-\gamma} = \frac{\gamma B_0}{1-\gamma} + \frac{(\mu - r^b)^2}{2\gamma\sigma^2} + r^b - B_0 \tag{D.18}$$

Lastly, solving for B_0 one can determine the missing parameter

$$B_0 = \frac{1}{2} \left[\rho + \gamma + 1 - (1-\gamma) r^b - \frac{1-\gamma}{\gamma} \frac{(\mu - r^b)^2}{2\sigma^2} \right] \square \tag{D.19}$$